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The constant elasticity of variance (CEV) model and the Legendre transform–dual solution for annuity contracts

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Abstract

The paper focuses on the constant elasticity of variance (CEV) model for studying a defined-contribution pension plan where benefits are paid by annuity. It also presents the process by which the Legendre transform and dual theory can be applied to find an optimal investment policy for a participant's whole life in the pension plan. Finally, it reveals two explicit solutions for the logarithm utility function in two different periods (before and after retirement). Hence, the optimal investment strategies in the two periods are obtained.

Classification: IE13; G23

Keywords: Defined-contribution pension plan; Stochastic optimal control; CEV model; Legendre transform; Optimal investment strategy

1. Introduction

The economic role of pension funds is considerable and well acknowledged in many countries. There are two different ways to manage a pension fund. One is that of defined-benefit plans, where benefits are defined in advance by the sponsor and contributions are initially set and subsequently adjusted so as to maintain the fund in balance. The other is that of defined-contribution plans, where contributions are fixed and benefits depend on the returns on the funds portfolio.

So the optimal investment strategy for a pension fund has become an important subject in recent years. Stochastic optimal control as a classical tool is used in the portfolio field and has been extended to pension funds management (e.g., Boulier et al. (2001) and Devolder et al. (2003)).

The constant elasticity of variance (CEV) diffusion model with stochastic volatility is a natural extension of geometric Brownian motion. This was originally proposed by Cox (1975, 1996) and Cox and Ross (1976). Afterwards, it was usually applied to analyze the options and asset pricing formula, as was done by Beckers (1980), Emanuel and

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Macbeth (1982), and recently by Davydov and Linetsky (2001) and Basu and Samanta (2001). However, in the last few years, the CEV model began to be applied in optimal investment research, as was done by Darius and Sircar (2003) in their working paper.

By using the method of stochastic optimal control, a non-linear partial differential equation (the Bellman equation) for the value function of the optimization problem will be derived. It is very hard to produce a closed-form solution, in the common sense, of this equation, especially under a CEV risk price process. But in some circumstances, the primary problem can be changed into a dual problem by applying a Legendre transform. Kramkov and Schachermayer (1999) stated and proved an existence and uniqueness theorem for the optimal investment strategy, and the relation between the primary problem and the dual problem. Choulli and Hurd (2001) and Jonsson and Sircar (2002) analyzed the Merton problem using the dual theory.

As far as we know, the application of the CEV model, the Legendre transform and the dual theory, used together, to a pension fund has not been reported in academic articles. Our research deals with the pension fund management issue in a continuous-time framework and focuses on defined-contribution plans where a guarantee is given for the benefits. We assume that the dynamics of stock price movements conforms to the constant elasticity of variance (CEV) diffusion process, not to the classical geometric Brownian motion. Consequently, a complicated non-linear partial differential equation (the Bellman equation) is derived by using the methods of stochastic optimal control. But it is difficult to find explicit solutions. So we transform the primary problem into the dual problem by applying a Legendre transform. We try to find the explicit solutions of the dual problem and the optimal investment strategies for the different periods (before and after retirement). Finally, we obtain two explicit solutions and the optimal investment strategies in two periods for the logarithm utility function. And at the same time, we compare our result with Merton's.

The paper is structured as follows. In Section 2, we introduce the CEV model, the Legendre transform and the dual problem. In Section 3, we establish the mathematical financial model for two phases: before retirement and after retirement. We devote Section 4 to finding the optimal portfolio with stochastic control theory and the Legendre transform. The effect of introducing future contributions before retirement is discussed in Section 5. Finally, we draw some conclusions.

2. Theory background

2.1. The constant elasticity of variance (CEV) model

The CEV model describes the dynamics of stock price movements using the form

$$\mathrm{d}s_t/s_t = \mu \mathrm{d}t + k s_t^{\gamma} \mathrm{d}w_t, \tag{2.1}$$

where s_t is the stock price, w is a standard Brownian motion, μ is the instantaneous expected return on stock over the interval $[t, t + \triangle t]$. The elasticity parameter (γ) was originally negative, but was extended to include positive values by Emanuel and Macbeth (1982). From (2.1) the variance of the stocks return $(\sigma_t)^2 = (k s_t^{\gamma})^2$.

2.2. The Legendre transform

Definition 2.1. Let $f : R^n \to R$ be a convex function; for $z > 0$, define the Legendre transform

$$L(z) = \max_x \{f(x) - zx\}. \tag{2.2}$$

The function $L(z)$ is called the Legendre dual of the function $f(x)$.

If $f(x)$ is strictly convex, the maximum in the above equation will be attained at just one point, which we denote by x_0. It is attained at the unique solution to the first-order condition

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} - z = 0. \tag{2.3}$$

So we may write

$$L(z) = f(x_0) - z x_0. \tag{2.4}$$

Example 1. Suppose that $f(x) = \ln x$. If we want to maximize $\ln x - zx$, then $x_0 = \frac{1}{z}$; thus $L(z) = -\ln z - 1$.