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Ruin probabilities in the discrete time renewal risk model☆

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Abstract

In this paper, we study the discrete time renewal risk model, an extension to Gerber's compound binomial model. Under the framework of this extension, we study the aggregate claim amount process and both finite-time and infinite-time ruin probabilities. For completeness, we derive an upper bound and an asymptotic expression for the infinite-time ruin probabilities in this risk model. Also, we demonstrate that the proposed extension can be used to approximate the continuous time renewal risk model (also known as the *Sparre Andersen risk model*) as Gerber's compound binomial model has been proposed as a discrete-time version of the classical compound Poisson risk model. This allows us to derive both numerical upper and lower bounds for the infinite-time ruin probabilities defined in the continuous time risk model from their equivalents under the discrete time renewal risk model. Finally, the numerical algorithm proposed to compute infinite-time ruin probabilities in the discrete time renewal risk model is also applied in some of its extensions.

Keywords: Ruin theory; Renewal risk model; Ruin probabilities; Upper bound; Numerical approximation; Variable premiums

1. Introduction

In this paper, we consider the discrete time renewal risk model, an extension to Gerber's compound binomial model, which has been studied in the actuarial literature by Pavlova and Willmot (2004) and Li (2005) among others. Under the assumptions of the compound binomial model, interclaim times have a geometric distribution. In the discrete time renewal risk model, we rather assume that the claim arrivals are governed by a (general) discrete time renewal process. Clearly, this extension for discrete time risk models is similar to the generalization of the classical compound Poisson risk model to the continuous time renewal risk model also known as the Sparre Anderson risk model (see e.g. Rolski et al., 1999).

In the discrete time renewal risk model, the claim number process $\underline{N} = \{N_k, k \in \mathbb{N}^+\}$ is a renewal process with interclaim times $\{T_j, j \in \mathbb{N}^+\}$ where $\{T_j, j \in \mathbb{N}^+\}$ is a sequence of independent and strictly positive integer-valued r.v.'s. $\underline{N}$ is called an *ordinary renewal process* if the r.v.'s $\{T_j, j \in \mathbb{N}^+\}$ are identically distributed with probability mass function (p.m.f.) f_T and cumulative distribution function (c.d.f.) $F_T(x) = 1 - \bar{F}_T(x)$. Letting the r.v. T_1 have the same distribution as the r.v.'s $T_2, T_3, \ldots$ is equivalent to assuming that a renewal has just occurred at time 0. If instead we

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keep the same assumptions but assume that the distribution of T_1 differs from that of $T_2, T_3, \ldots$, $\underline{N}$ is referred to as a *delayed renewal process*. A common choice for the distribution of T_1 is the equilibrium distribution for which $\underline{N}$ is called a *stationary renewal process*. In the literature, there are two possible definitions for the equilibrium distribution (see Hu et al., 2003). As in Pavlova and Willmot (2004), we consider in this paper the following equilibrium distribution for the r.v. T_1,

$$f_{T_1}^{\mathrm{e}}(k) = \frac{\bar{F}_T(k-1)}{E[T]}, \qquad k \in \mathbb{N}^+ \tag{1}$$

when $\underline{N}$ is a *stationary renewal process*.

The individual claim amount r.v.'s $\{X_j, j \in \mathbb{N}^+\}$, where X_j corresponds to the amount of the jth claim, are assumed to be a sequence of strictly positive, independent and identically distributed (i.i.d.) r.v.'s with p.m.f. f_X and c.d.f. $F_X(x) = 1 - \bar{F}_X(x)$. Moreover, we assume that the r.v.'s $T_1, T_2, \ldots$ and $X_1, X_2, \ldots$ are mutually independent. The total claim amount process $\underline{S} = \{S_k, k \in \mathbb{N}\}$ is defined as $S_k = \sum_{j=1}^{N_k} X_j$, where $\sum_a^b$ equals 0 if $b < a$.

Finally, we define the surplus process $\underline{U} = \{U_k, k \in \mathbb{N}\}$ as $U_0 = u$ and $U_k = u + ck - S_k$ for $k \in \mathbb{N}^+$ where $u (u \in \mathbb{N})$ is the initial surplus level and $c (c \in \mathbb{N}^+)$ is the level premium received per period. For discrete time risk models, there are two distinct definitions of the time of ruin in the literature. In the first one, it is assumed that ruin occurs when the surplus falls below zero while in the second, ruin is assumed to occur when the surplus falls below or at the level 0 (see e.g. Shiu, 1989 and Gerber, 1988). In this paper, we consider only the first one, namely, $\tau = \inf_{k \in \mathbb{N}^+}\{k, U_k < 0\}$ is defined as the time of ruin associated to $\underline{U}$ with $\tau = \infty$ if $U_k \geq 0$ for all $k \in \mathbb{N}$ (i.e. ruin does not occur). Let $\psi(u, n) = E[1_{\{\tau \leq n\}}]$ and $\psi(u) = E[1_{\{\tau < \infty\}}]$ be the finite-time and infinite-time ruin probabilities where the indicator function $1_A = 1$ if A is true and 0 otherwise. Their complements, the finite-time and infinite-time non-ruin probabilities, are respectively denoted by $\phi(u, n)$ and $\phi(u)$. To ensure that $\psi(u)$ in the discrete time renewal risk model goes to 0 as $u \to \infty$, the premium rate is such that

$$cE[T] > E[X]. \tag{2}$$

The paper is structured as follows: in Sections 2 and 3, we study the aggregate claim amount process and the finite-time and infinite-time ruin probabilities in the discrete time renewal risk model. In Section 4, an upper bound and an asymptotic expression for the infinite-time ruin probabilities in the discrete time renewal risk model are provided for completeness. In Section 5, we derive both numerical upper and lower bounds for the infinite-time ruin probabilities in the continuous time renewal risk model through the infinite-time ruin probabilities in the discrete time renewal risk model. Finally, in Section 6, we briefly present some additional results under the framework of the discrete time renewal risk model and in some of its extensions.

2. Aggregate claim amount process

In this section, we study the aggregate claim amount process in the discrete time renewal risk model. In this paper, most results will be derived in the ordinary case since the delayed case can easily be deduced from the ordinary one by noting that the claim number process just beginning at time T_1 is an ordinary renewal process.

When $\underline{N}$ is an ordinary renewal process, it is well known that $E[S_k] = E[X]E[N_k]$ where $E[N_k] = \sum_{j=1}^{\infty} F_T^{*j}(k)$ is the solution to the discrete time renewal equation

$$E[N_k] = F_T(k) + \sum_{j=1}^{k} E[N_{k-j}] f_T(j), \tag{3}$$

for $k \in \mathbb{N}^+$ (see Rolski et al., 1999) assuming $E[N_0] = 0$. When $\underline{N}$ is a stationary renewal process, we have $E[S_k^{\mathrm{st}}] = E[X]E[N_k^{\mathrm{st}}]$ where, in this case, $E[N_k^{\mathrm{st}}] = k/E[T]$ is obtained by induction. Clearly, $E[N_1^{\mathrm{st}}] = f_{T_1}^{\mathrm{e}}(1) = 1/E[T]$. Then, suppose that $E[N_j^{\mathrm{st}}] = j/E[T]$ holds for $j \in \{2, \ldots, k-1\}$. By first conditioning on the r.v. T_1 with p.m.f. (1), one obtains

$$E[N_k^{\mathrm{st}}] = \sum_{i=1}^{k} f_{T_1}^{\mathrm{e}}(i) + \sum_{i=1}^{k} f_{T_1}^{\mathrm{e}}(i) E[N_{k-i}]. \tag{4}$$