Int. J. Production Economics ∎ (∎∎∎∎) ∎∎∎–∎∎∎

Contents lists available at ScienceDirect

Int. J. Production Economics

journal homepage: www.elsevier.com/locate/ijpe

Cumulative staircase considerations for dynamic lotsizing when backlogging is allowed

Robert W. Grubbström [a,b,*]

[a] Linköping Institute of Technology, SE-581 83 Linköping, Sweden
[b] Mediterranean Institute for Advanced Studies, Primorski tehnoloski park, SI-5290 Šempeter pri Gorici, Slovenia

ARTICLE INFO

Article history:
Received 23 October 2012
Accepted 22 May 2013

Keywords:
Dynamic lotsizing
EPQ
Finite production rate
Backlog
Shortages
Net present value
Binary approach

ABSTRACT

The *dynamic lotsizing problem* concerns the determination of optimal batch quantities, when given required amounts appear at discrete points in time. The standard formulation assumes that no shortages are allowed and that replenishments are made instantaneously.

For the case when no shortage is allowed, previously it has been demonstrated that the *inner-corner condition* for an optimal production plan in continuous time reduces the number of possible replenishment times to a finite set of given points at which either a replenishment is made, or not. The problem is thus turned into choosing from a set of zero/one decisions with 2^{n-1} alternatives, of which at least one solution must be optimal, where n is the number of requirement events. Recently, the instantaneous replenishment assumption has been replaced by allowing for a finite production rate, which turned the inner-corner condition into a condition of tangency between the cumulative demand staircase and cumulative production.

In this paper we investigate relationships between optimal cumulative production and cumulative demand, when backlogging is permitted. The production rate is assumed constant and cumulative production will then be a set of consecutive ramps. Cumulative demand is a given staircase function. The net present value (NPV) principle is applied, assuming a fixed setup cost for each ramp, a unit production cost for each item produced and a unit revenue for each item sold at the time it is delivered.

Among other results, it is shown that optimal cumulative production necessarily intersects the demand staircase. Instead of having 2^{n-1} production staircases as candidates for optimality, there are 2^{n-1} production structures as candidates. These are made up of sequences of batches, of which the set of batches may be optimised individually. Also is shown that the NPV of each batch has a unique timing maximum and behaves initially in a concave way and ends as convex.

Results for the average cost approach are obtained from a zeroth/first order approximation of the objective function (NPV).

1. Introduction

The dynamic lotsizing problem is an extension of the classical EOQ lotsizing problem (Harris, 1913). This extension was published by Wagner and Whitin (1958). Requirements (demand) to be satisfied are distributed over time in given amounts, and the problem is to decide how much to produce at different times in order to optimise an objective function, in the simplest case minimising the total sum of holding costs and setup costs.

Originally, production was assumed to take place instantaneously, but Hill (1997) relaxed this assumption allowing for a finite production rate.

The timing framework for the problem is usually a discrete-time scale, but it has been advocated by the current author that a continuous-time scale is a superior approach, including the discrete time as a special case.

The objective function has traditionally involved the average cost (AC) approach. Against this stands the more sophisticated net present value (NPV) principle, applying an interest rate for modelling capital costs, and the AC objective is normally obtained, when the NPV is approximated as a first/zeroth-order expansion in this interest rate. Applying NPV to lotsizing problems was probably first published by Hadley (1964), but since then there have been several contributions, and the interest for using this principle appears to be increasing (Trippi and Levin, 1974, Grubbström, 1980, Kim et al., 1986, Teunter and van der Laan, 2002, Beullens and Janssens, 2011).

The *inner-corner condition* is a necessary condition for optimality, when formulating requirements and production as cumulative

* Correspondence address: Linköping Institute of Technology, SE-581 83 Linköping, Sweden. Tel.: +46 705 127077.
E-mail address: robert@grubbstrom.com

0925-5273/$ - see front matter © 2013 Published by Elsevier B.V.
http://dx.doi.org/10.1016/j.ijpe.2013.05.027

functions of time. When production takes place instantaneously, both of these functions become staircase functions, and this necessary condition means that the optimal production staircase fits into the given demand staircase at inner corners as is illustrated in Fig. 1. This condition is valid when either the AC or the NPV principle is followed. It is also valid for a general assembly system (Grubbström et al., 2010, Grubbström and Tang, 2012).

The inner-corner condition limits the set of possible optimal solutions to a finite set of production staircases, since at each inner corner there is either a contact, or no contact. If there are n steps in the demand staircase, the number of possible solutions become 2^{n-1}, observing that there is always a contact at the very first inner corner. The dynamic lotsizing problem is then turned into a binary problem, at each demand event either production takes place or there is no production. This binary property was early recognised by Veinott Jr., (1969), although neither a continuous time scale, nor the NPV objective appear to have been applied to the problem formulation at that time.

As shown in a recent paper (Grubbström, 2012), when the production rate is assumed to be finite, rather than that production takes place instantaneously, then inner-corners no longer appear. Instead a condition of tangency applies, and cumulative production now built up of a set of ramps either touch an upper corner of the demand staircase, or not, see Fig. 2. The binary property thus is still valid, but it is now based on a different geometrical structure.

Hitherto, the problem formulation has involved the assumption that all demand must be satisfied and that therefore no shortages are allowed. This assumption was relaxed by Song and Chan (2005) in an article allowing for backlogs to be possible. The timing framework was there assumed to be discrete, and cumulative demand and production functions had not been introduced.

The problem treated in this paper is a follow-up of Grubbström (2012) allowing for shortages to be possible at the cost of delaying the in-payments for backlogged items. Other penalties for negative consequences from shortages (such as badwill) are not considered here. Neither is the possibility of lost sales taken into consideration.

Cumulative functions are considered and economic consequences concern sales revenues and costs for setups and production. Basically an NPV approach is followed, and the corresponding AC results are compared with its first/zeroth order approximations. It is shown that the "inner-corner/tangency" condition now must be replaced by a weaker binary condition. Still there are 2^{n-1} possible structures to be chosen from, but these are not immediately recognised as to the

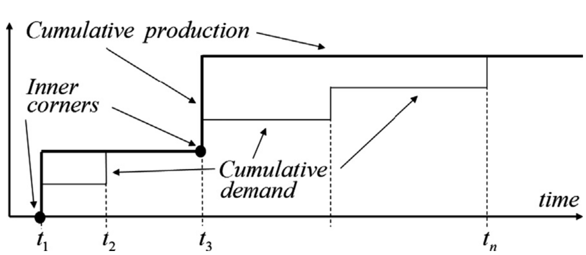

Fig. 1. The inner-corner condition as a necessary provision for optimality.

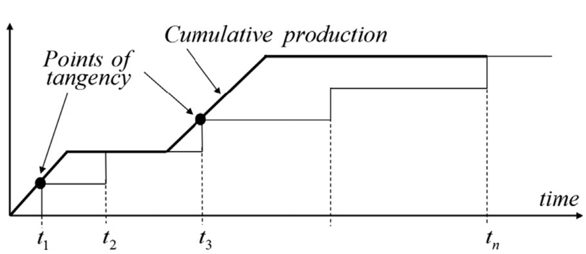

Fig. 2. The "tangency" condition as a necessary prerequisite for optimality when no shortages are allowed.

timing of the individual production steps/ramps. Instead, each part of the cumulative production function involves an individual local optimisation.

A production structure eligible for optimality is thus a set of consecutive production ramps beginning and ending at points in time defined by the given requirements (the demand events) and covering all steps in the cumulative demand staircase. Each batch belonging to a structure may be optimised on its own, irrespective of which structure that currently is considered. Since there are 2^{n-1} structures, and only $n(n+1)/2$ different batches, in general, a certain batch will belong to more than one structure. From a computational point of view, each batch can therefore be optimised on its own (by a suitably choice of its timing), and the dynamic lotsizing problem will then concern the question of matching the sequence of optimised batches in order to maximise the objective function in the NPV case, or minimise total costs in the AC case. The number of possible different batch sizes is seen from: one opportunity to form a batch covering all n steps (All-At-Once), two opportunities to form a batch covering $(n-1)$ steps, …, and n opportunities to form a batch covering just one step (Lot-For-Lot), together accounting for the arithmetic sum $\sum_{i=1}^{n} i = n(n+1)/2$. It might be interesting to note that for very small values of n, the number of batch sizes is larger than the number of structures, i.e. $2^{n-1} < n(n+1)/2$, for $2 \le n \le 4$.

In Section 2 we first show that the segments of the optimal production structure start and end on levels belonging to the horizontal steps of the demand staircase. This is followed by a subsection treating conditions for optimal production segments (ramps, batches). It is shown that each batch has a unique optimal start time and that the optimal ramp necessarily intersects the demand staircase, when backlogging is allowed. Optimising the batches individually, provides a pool of, in total, $n(n+1)/2$ batches to be chosen from when selecting the production structure maximising the overall NPV. Section 3 includes a numerical example illustrating our findings. This is followed by a short concluding section summarising our results.

2. Introducing the possibility of shortages

Table 1 lists the main notation to be used in the following. Additional notation is introduced when the need appears.

2.1. Necessary condition for optimal production structure

Our starting point consists of two geometrical structures, on the one hand the given cumulative demand (requirements) staircase, on the other the cumulative production function made up of a set of ramps, each having a slope of q, and joined by horizontal steps. Fig. 3 illustrates the two functions.

Table 1
Main notation.

n	Given number of requirement events
t_i	Given time at which the ith requirement appears, $i=1, 2, \ldots, n$
D_i	Given size of requirement at time t_i, $i=1, 2, \ldots, n$
$\overline{D}_i = \sum_{j=1}^{i} D_j$	Cumulative requirements immediately after time t_i
p	Unit sales price (in-payment)
c	Unit production cost (out-payment)
K	Setup (ordering) cost for producing one batch (out-payment)
h	Inventory holding cost per unit and time unit
b	Backlog cost per unit and time unit
ρ	Continuous interest rate, per time unit
Q	Batch size
q	Finite production rate, units per time unit
β	Parameter indicating if setup costs are allocated to the beginning of production intervals ($\beta=0$) or to interval ends ($\beta=1$)