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Exact solutions to the supply chain equations for arbitrary, time-dependent demands

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ABSTRACT

We study the impact on inventory of an unexpected, non-linear, time-dependent demand and present the exact solutions over time to the supply chain equations without requiring any approximations. We begin by imposing a boundary condition of stability at infinity, from which we derive expressions for the estimated demand and the target work in progress when the demand is time-dependent. The resulting inventory equation is solved in terms of the Lambert modes with all of the demand non-linearities confined to the pre-shape function. The series solution is exact, and all terms are reasonably easy to calculate, so users can determine the inventory behavior to any desired precision. To illustrate, we solve the equations for a non-linear, quadratic time-dependence in the demand. For practical use, only a few terms in the series are required, a proposition illustrated by the For All Practical Purposes (FAPP) solution. While the paper provides a theoretical foundation, the result is decidedly practical: An accurate and reasonably easy-to-implement model that companies can use to analyze the impact of non-linear, time-dependent demands.

1. Introduction

Decreasing product life-cycles, rapidly evolving consumer tastes and the continual introduction of new products result in complex, time-dependent demands (Tang and Tang, 2002). For such volatile products, there may be little historical data on which to base estimates of the future demand and successful ordering strategies are much more difficult to develop (Warburton and Stratton, 2002). For companies with stable product lines, historic sales can be reasonable predictors of future demand, leading to effective inventory management of such products with a corresponding improvement in financial performance (Capkun et al., 2009).

We address this problem by providing a model of inventory control for arbitrary, non-linear, and time-dependent demands. We assume, as in much of the real world, that unexpected, non-stationary demand changes occur, and that companies must deal with those surprises without foreknowledge. We work in continuous system dynamics in the time domain (Warburton, 2004b) where the demand, inventory and work-in-process (WIP) levels are continuously observed and forecasts are continuously adjusted to reflect the most up-to-date information. Our goal is to provide analytical techniques to companies so that they can study several different models of possible future demand scenarios and develop appropriate ordering strategies.

The main contribution of this paper is the derivation an exact solution for the inventory behavior over time as it reacts to an unexpected change in demand and even for one that is non-linear. The solution is quite general and no approximations or simplifications are required. The solution is in terms of a series, in which all terms are easily calculated, so users can determine the inventory behavior to any desired precision.

We derive the expression for the Target Work-in-Process when the demand is time-dependent and show that a stable inventory requires a perfectly known future demand, which is, in practice, unknown and must be estimated. However, even if the demand is incorrectly forecast, the solution is still exact, just not guaranteed to be stable. We demonstrate this by exactly solving the equations for an imperfectly known, non-linear demand and show that the inventory does indeed diverge.

To illustrate the method, we solve the equations for a non-linear demand with a quadratic time-dependence. While the paper provides a theoretical foundation, the contribution is decidedly practical: An accurate and reasonably easy-to-implement model that companies can use to develop ordering strategies for non-linear, time-dependent demands.

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The structure of the paper is as follows. The next section reviews the literature applicable to time-dependent demand models. The inventory and ordering equations are introduced and then, using the boundary condition at infinity, we derive explicit expressions for the estimated demand and the Target Work-in-Process (WIP) when the demand is time-dependent (Section 2). In Section 3, we derive the exact analytical solutions to the combined inventory and ordering equations. To illustrate the method, in Section 4 we derive the exact inventory solution when the demand has a non-linear, quadratic time-dependence. In Section 5, we provide three practical applications: the exact solution when the demand is mis-estimated; a practical approach, the For All Practical Purposes (FAPP) solution, which is easy to calculate and reasonably accurate; and, the impact of adding random noise to a quadratic, time dependence in the demand. Finally, we provide some concluding remarks in Section 6.

1.1. Relevant literature

Our model is based on the continuous-time form of the inventory and ordering Warburton, 2004a, 2004b. Childerhouse and Towill (2000) established the basis for continuous models from which a number of important features of supply chains have already emerged, including inventory stability properties (Warburton et al., 2004) and resonances in multi-echelon supply chains (Hodgson and Warburton, 2009).

Continuous-time models are consistent with discrete-time models, which are typically analyzed with system dynamics. Disney and Lambrecht (2008) provided a comprehensive account of the state-of-the-art and an extensive literature survey. Disney et al. (2006) demonstrated that management insights gained from the continuous and discrete time domains are very similar, and concluded that one may use either time domain in an analysis. This is particularly relevant because questions that are difficult to analyze in one time domain may be easier in the other and, while the exact results may differ slightly, the qualitative nature of the management conclusions are essentially equivalent (Warburton and Disney, 2007). Therefore, the insights from this analysis should translate into the discrete domain.

We use an ordering policy where the aim is to keep the inventory as close as possible to some target base-stock level, which has proven to be a beneficial technique in planning, inventory control, and forecasting (Towill, 1982). The same policy has been the subject of extensive analysis in both the continuous and discrete domains (Dejonckhere et al., 2004). For example, Nielsen and Nielsen (2008) show that small changes in base variables, such as skills, the customer base, or the work in process may have a major influence on profit, and may be impossible to predict without a dynamic model.

Our goal is to provide exact solutions to the supply chain equations when there is a time-dependent demand. Usually, such models are complex and are not amenable to exact solutions. For example, Aviv (2003) modeled a demand that evolved according to a vector autoregressive time series, and proposed an adaptive inventory replenishment policy for the supply chain members. Aviv (2007) later proposed a unified time-series framework for forecasting and inventory control, with the critical observations that supply chains are rich with demand information, that forecasting must cope with a variety of demand characteristics, and that models need to be continually revised over time, confirming the importance of the domain we have chosen to study. Earlier, Aviv (2002) studied an auto-regressive time series demand, which is valuable because one can study more realistic (i.e., non-i.i.d.) demands. Other approaches include those of Diponegoro and Sarker (2007), who formulated an integer, non-linear programming problem, and Moon et al. (2006), who used an adaptive genetic algorithm for advanced planning. These studies illustrate the need for analytical solutions when the demand is time-dependent.

We also study the impact of noise on a non-linear demand because sometimes the data are so noisy, or the trend is so erratic, that even a linear trend is not accurate (Roberts, 1982). Gardner and McKenzie (1985) introduced a damped trend procedure that works well in these situations. In a more general theoretical framework, Levi et al. (2005) addressed the problem of finding computationally efficient, and provably good, inventory control policies with correlated and non-stationary (time-dependent) stochastic demands. We improve on these studies by providing analytical solutions from which the properties are derivable.

Continuous mathematics is also employed in differential game models (El Ouardighi et al., 2008), where the goal is to develop strategies that are optimal for both the operational and marketing departments. Marketing's objective is to maximize the discounted flow of revenues minus quadratic advertising costs over an infinite time horizon. Our model uses an entirely different approach because we do not need to assume any relationships or conduct an optimality argument as the inventory and ordering equations are well-established.

Another important paradigm relevant to complex demand structures is dynamic programming, which is effective in characterizing the optimal policy. While the models are often complex, the optimal policies are sometimes relatively simple (Lingxiu and Lee, 2003; Zipkin, 2000). However, a significant problem is that correlated and non-stationary demands cause the state space to grow exponentially—the curse of dimensionality. Also, when the demand varies significantly, promising policies may perform poorly (Levi et al., 2007). These approaches provide some general guidance but not a general solution.

We include the standard WIP terms, the importance of which was established by Enns (2001) and Selcuk et al. (2006). Ramdas and Spekman (2000) established that WIP tracking seems to be a strategic necessity. Naim and Towill (1995) originally introduced WIP into a supply chain model, followed by Disney et al. (1997) whose model assumed a time-independent demand and, therefore, a constant WIP. We use the same structure for our ordering policy, but generalize it to include non-linear time-dependencies.

Many papers assume that the Target WIP is a constant, which makes sense when the demand is i.i.d. and the mean demand is known. However, when the demand is time-dependent, so are the WIP and Target WIP terms. Dejonckhere et al. (2004) treat the Target WIP as dynamic and while their expression for the Target WIP looks reasonable, it was simply assumed and not formally justified. We will provide a different expression for the Target WIP, one that guarantees stability. We also explain why the Dejonckhere et al. (2004) expression works reasonably well in practice.

When the demand varies, long lead times impose high costs due to rising inventory, safety stocks, and WIP. Pahl et al. (2007) provided an in-depth discussion of the state-of-the art. Goncalves (2006) provided insight into the importance of time-dependent demands and, in particular, oscillations, which affect supply chain instability. When the demand varies over time, our analytical model allows the study of both stability and long lead times.

A common supply chain issue facing companies is the so-called Bullwhip Effect, in which forecasts are based on the demand for intermediate products, and not on the true demand for the product (Lee et al., 1997b, 1997a; Chen et al., 2000). Metters (1997) quantified that it can be very costly and, interestingly, the real world may have to cope with bullwhip amplification as high as 20:1 (Holmstrom, 1997). For time-dependent demands, the definition of the Bullwhip Effect is problematic because the definitions usually assume time-independent variances. Our approach is to present the evolution of the inventory, a model that is more suited to a demand that evolves over time.