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A branch & bound algorithm for cutting and packing irregularly shaped pieces

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ABSTRACT

Cutting and packing problems involving irregular shapes, usually known as Nesting Problems, are common in industries ranging from clothing and footwear to furniture and shipbuilding. Research publications on these problems are relatively scarce compared with other cutting and packing problems with rectangular shapes, and are focused mostly on heuristic approaches. In this paper we make a systematic study of the problem and develop an exact Branch & Bound Algorithm. The initial existing mixed integer formulations are reviewed, tested and used as a starting point to develop a new and more efficient formulation. We also study several branching strategies, lower bounds and procedures for fixing variables, reducing the size of the problem to be solved at each node. An extensive computational study allows us first to determine the best strategies to be used in the Branch & Bound Algorithm and then to explore its performance and limits. The results show that the algorithm is able to solve instances of up to 16 pieces to optimality.

1. Introduction

Nesting problems are two-dimensional cutting and packing problems involving irregular shapes. These problems arise in a wide variety of industries like garment manufacturing, sheet metal cutting, furniture making and shoe manufacturing. The wide range of applications implies many different variants of the problem. The placement areas into which the pieces have to be packed or cut can be rectangular, as in the case of materials provided in rolls, or can have irregular shapes, as in the case of leather hides for making shoes. This placement area can have a uniform quality or different qualities depending on the region, including sometimes defective parts that cannot be used for pieces. The pieces to be cut can be described as polygons, convex or not, or can include curved edges. Depending on the real application, the pieces can be rotated freely, at specific angles only (90°, 180°, …), or not rotated at all. There may also be different objectives, usually involving the minimization of the area required for cutting the pieces or the maximization of the value of the pieces cut. Using the typology proposed by Wäscher et al. (2007), nesting problems, in general, are open-dimension problems with irregular pieces.

In this paper we consider a nesting problem in which we have to arrange a set of two-dimensional irregular shapes without overlapping in a rectangular stock sheet of a fixed width, where the objective is to minimize the required length. We will consider the pieces to be described as polygons, not necessarily convex, which cannot be rotated.

The main difficulty of nesting problems is to ensure that the pieces have a non-overlapping configuration. This question has been studied extensively in recent years and there are several approaches which determine when two polygons overlap. Bennell and Oliveira (2008) give a tutorial on the different approaches which study the geometry of nesting problems. The problem is NP-complete and as a result solution methodologies predominantly utilize heuristics.

Pixel/Raster methods transform the continuous stock sheet into a discrete grid represented by a matrix, and the position of each piece adds a given coded value to each matrix element. Identifying possible overlapping comes down to checking the matrix values. There are three known codification schemes the first one proposed by Segenreich and Braga (1986), the second one by Oliveira and Ferreira (1993), and the last one by Babu and Babu (2001).

When pieces are represented (or approximated) using polygons, there are tests for edge intersection and for point inclusion that can be used for identifying overlapping (Konopasek, 1981; Ferreira et al., 1998). The most widely used tool for checking whether two polygons overlap is the No-Fit Polygon (NFP). It can be used, along with the vector difference of the position of the two polygons, to determine whether these polygons overlap, touch or are separated, by conducting a simple test to identify whether the resulting vector is inside the NFP. There are three approaches to generating the NFP the orbiting algorithm by Mahadevan (1984), improved by Burke et al. (2007); the Minkowski sums used by

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Milenkovic et al. (1991) and Bennell and Dowsland (2001); and the decomposition into star-shaped polygons (Daniels et al., 1994) or convex polygons (Watson and Tobias, 1999; Agarwal et al., 2002).

A more general tool that generalizes the NFP is the *Phi-function*. Its purpose is to represent all the mutual positions of two polygons. The *Phi-functions* for cutting and packing were conceived of and applied by Stoyan et al. (2001, 2004), Scheithauer et al. (2005). Bennell et al. (2010) provide a good explanation of how the *Phi-functions* can be built, as well as some applications for them.

Many different heuristic and metaheuristic approaches have been proposed for solving nesting problems. Simple heuristic rules are used for building a solution step by step, placing one piece at a time, using different placement procedures and different piece sequences. Metaheuristic procedures are used for working with complete solutions and modifying them iteratively in order to find improvements.

Very few integer linear programming formulations have been proposed to date. One of them appears in the Simulated Annealing Algorithm proposed by Gomes and Oliveira (2006). In their algorithm, they use a compaction phase in which they solve a linear program which is a relaxation of a mixed-integer formulation of the problem. A different approach, based on Daniels et al. (1994) and Li (1994), is proposed by Fischetti and Luzzi (2009), introducing the concept of *slices* which, for each pair of pieces, partitions the region outside the corresponding NFP, that is the region in which the second piece can be placed without overlapping the first piece.

The remainder of this paper is organized into five sections. In Section 2 we describe the elements of the problem in detail, including the NFP, as it will be the basic tool for developing our formulation. Section 3 is devoted to the mixed integer linear formulation. We review the Gomes and Oliveira (2006) approach and introduce two new formulations, using the main ideas of the Fischetti and Luzzi (2009) model. In Section 4 we present our own Branch and Bound algorithm, describing several branching strategies, lower bounds and variable-reduction procedures. Section 5 contains an extensive computational experiment, using known and new test instances, which allows us to compare the proposed alternatives and the overall efficiency of the algorithm. Finally, in Section 6, we draw some conclusions and outline future work.

2. The problem

Let $P=\{p_1,\dots,p_N\}$ be the set of pieces to arrange in the strip. We consider the reference point of each piece to be the bottom-left corner of the enclosing rectangle. We denote the coordinates of the reference point of piece p_i by (x_i,y_i), its length by l_i and its width by w_i (see Fig. 1). The dimensions of the strip are its width W (fixed) and its length L (to be determined). We consider the bottom-left corner of the strip to be placed at the origin.

For each pair of pieces, p_i and p_j, the *No-Fit Polygon*, NFP_{ij}, is the region in which the reference point of piece p_j cannot be placed because it would overlap piece p_i. The feasible zone to place p_j with respect to p_i, outside NFP_{ij}, is a non-convex polygon or it could be unconnected. Fig. 2 shows a simple case in which piece i is a square and piece j a triangle. To build NFP_{ij}, the reference point of piece i is placed at the origin and the reference point of piece j slides around piece i in such a way that one point of piece j is always touching the border of piece i. The left-hand side of Fig. 2 shows several positions of piece j moving around piece i. The right-hand side of the figure shows NFP_{ij}, the forbidden region for the reference point of piece j, relative to piece i, if no overlapping is allowed.

When one or both polygons are non-convex, building the NFP is more complex. Fig. 3, taken from Bennell and Oliveira (2008), shows more complicated cases. In Fig. 3(a), piece B has some feasible positions within the concavity of A and therefore NFP_{AB} includes a small triangle of feasible placements for the reference point of B. In Fig. 3(b), the width of B fits exactly into the concavity of A and its feasible positions in the concavity produce a segment of feasible positions for the reference point of B in NFP_{AB}. In Fig. 3(c), there is exactly one position in which B fits into the concavity of A and then NFP_{AB} includes a single feasible point for the reference point of B. The methods mentioned in the previous section are powerful enough for building the NFP in all possible cases. In this paper we assume that for each pair of pieces i, j, NFP_{ij} is given by a polygon plus, if necessary, some points (as in Fig. 3(c)), some segments (as in Fig. 3(b)) or some enclosed polygons (as in Fig. 3(a)).

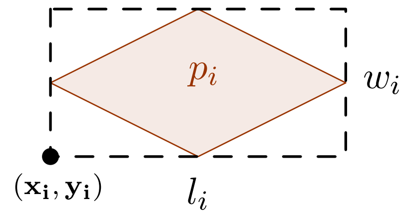

Fig. 1. Reference point of piece p_i.

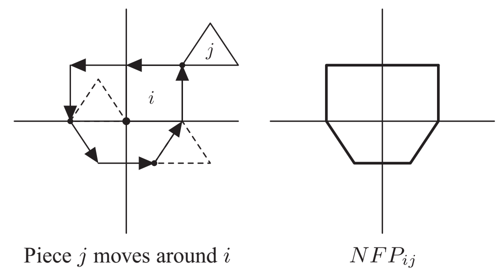

Fig. 2. Building the *NFP* of pieces *i* and *j*.

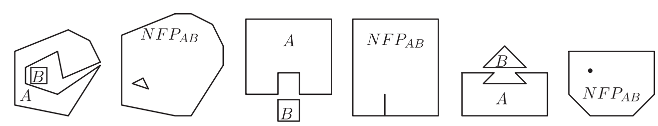

Fig. 3. Special cases of NFP when non-convex pieces are involved.

3. Mixed integer formulations

In this section we will first describe the formulation used by Gomes and Oliveira (2006) and then our two proposals, based on the ideas of Fischetti and Luzzi (2009). In all cases, the objective function will be the minimization of L, the strip length required to accommodate all the pieces without overlapping. Also, all formulations contain two types of constraints those preventing the pieces from exceeding the dimensions of the strip and those forbidding the pieces from overlapping. The differences between formulations lie in the way these constraints are defined.

3.1. Formulation GO (Gomes and Oliveira)

Let us consider the simple example in Fig. 4. Pieces p_i and p_j are rectangles, and then NFP_{ij} is a rectangle. Associated with each edge of NFP_{ij}, a binary variable v_{ijk} is defined. Variable v_{ijk} takes value 1 if the reference points of piece j and piece i are on different sides of the line defined by the kth edge of NFP_{ij}, or the reference point of piece j even being on that line, otherwise it