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Analytical evaluation of lead-time demand in polytree supply chains with uncertain demand, lead-time and inter-demand time

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ABSTRACT

In a stochastic supply chain network knowing the exact distribution of lead-time demand (LTD) for each node allows determining inventory policy parameters, service levels, and optimal costs. To determine the LTD distributions a stochastic analytical model has been developed for a multi-echelon supply chain organised into an arbitrary polytree network, where the variables: demand quantity, lead-time, and inter-demand time are uncertain and can follow any probability distribution. The model is based on a conventional *nQ* batch policy for each node with discrete time and continuous demand. Any network can be broken down into four basic elements which represent root and leaf node topologies. The model derives key convolutions for the four basic elements and then follows the same technique for evaluating resulting LTD distributions for each node in the network. The paper presents the formulation in seven steps that can be used to obtain exact LTD distributions either in the two dimensional form or as a compound one-dimensional LTD distribution per variable lead-time for each node in any polytree supply chain. The LTD distributions can be calculated either analytically for smaller networks or using numerical integration for larger networks. The validation of the developed framework has been conducted by comparing the obtained formalism-based LTD distribution for two test networks against simulation. The pairs of simulated and formalism-based distributions have been plotted against each other as well as subjected to various goodness-of-fit tests and error analysis. The results suggest that the formalism-based and simulated probability distributions are consistent.

1. Introduction

For stochastic demand the lead-time demand (LTD) distribution is an important factor that allows determining inventory policy parameters, service levels and optimal costs. In a stochastic supply chain network for the same reasons one needs to know the exact or approximate distribution of such demand for each node of the network.

The LTD distribution in a single node inventory system has been studied from different perspectives. Most of the research is conducted in the area of forecasting and is directed towards predicting the LTD distribution based on available, often fractured, data of demand per unit time (period). Most studies approximate the LTD distribution to known probability distributions such as Normal (Girlich, 1996), Geometric (Carlson, 1982), Erlang (Levén and Segerstedt, 2004), Lognormal (Tadikamalla, 1979), and other distributions, usually basing such approximations on particular distributions of demand or lead-time. For a one-to-one renewal process a general model was developed by Zipkin (1988) that uses phase-type distributions for demand and lead-time for modelling the LTD distribution. Often historical data is used to estimate first moments of the LTD distribution that are then used directly in inventory decision making. For general distributions of demand and lead-time Lau and Wang (1987) calculate the first four moments of the unit time demand and lead-time distribution to estimate the first four moments of LTD. A similar approach is used by Tang and Grubbström (2006), while applying the first and second moments to Normal and Gamma approximations to LTD they also use higher order moments to other distributions. A slightly different approach based on estimating only first two moments was developed by Shore (1999).

The applicability of various approximations to LTD have been analysed in many studies. Tadikamalla (1984) compares several approximations to the LTD distribution while Bagchi et al. (1984) present a classification of different forms of obtained compound LTD distributions derived from particular distributions of demand components and provide relevant references. More recently, detailed studies of some commonly used approximations to LTD have been conducted. The robustness of different LTD models has been examined by Rossetti and Ünlü (2011) who present a model for selecting a most suitable LTD distribution by relying on given LTD mean and variance. The authors have found that distribution

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selection rules have a great potential for modelling LTD. Earlier Lau and Lau (2003) investigated the robustness of Normal approximation to LTD and compared major studies related to the LTD distribution. The authors confirm earlier findings of Bagchi et al. (1986) that the importance of the LTD shape is generally underestimated so it is worthwhile to estimate an LTD-distribution's shape more accurately. In this regard Park (2007) examined two analytical approaches of deriving compound LTD distribution for a single stage and suggested which of them are better suitable for particular situations.

This paper looks at the determination of the LTD distribution from a similar perspective. If the distributions of demand and lead-time have already been estimated, e.g. generic forecast, moments based approximation, or advanced demand information (Gallego and Özer, 2001), can then the LTD distribution be determined exactly, for a general case, and for any node in any arbitrary network? One of the ways to achieve this is to conduct all the necessary convolutions with the original distributions. Some researchers employ convolutions for solving certain problems, but often use approximate methods for their evaluation rather than explicit expressions (e.g. Hayya et al., 2009). The precise form of such convolutions may not necessarily be apparent and no complete formalism for a general case and for a general network is immediately available. Yet, the LTD distribution can be relatively easily determined analytically for a single stage inventory system using direct convolutions or probability generating functions (McFadden, 1972; Bagchi et al., 1984). When the number of nodes increases the complications very quickly become intractable and approximate methods are used instead (Kiesmüller et al., 2004). Though such methods achieve their objectives, in many problems an analytical solution in explicit or general form for LTD distribution is preferential as it ensures a required level of accuracy.

In this paper it is assumed that the forms of the initial probability distributions, i.e. those of demand per unit time and lead-time, are already known and are independently and identically distributed (*iid*) random variables. Since for different inventories and for different echelons such distributions can range from normal to a long tail a framework, that is not limited to a particular process and allows input of either of known or generic (historical) distributions, was anticipated.

For a stochastic multi-echelon inventory system to the best of our knowledge no general analytical framework for determining LTD distribution in particular, or service levels in a broader perspective, has been proposed. This is despite the fact that a great number of the developed cost optimisation models starting from Clark and Scarf (1960) are based on the LTD distribution as on known input (e.g. Langenhoff and Zijm, 1990, reviews by Axsäter, 2003 and Gümüs and Güneri, 2007). There are fewer papers that estimate the LTD distribution by approximating it to particular distributions. For example, Houtum and Zijm (1991) have developed algorithms for LTD distribution for multistage systems that are based on incomplete convolutions of mixtures of Erlang distributions. In a broader range the existing multi-echelon inventory models, whether dealing directly with LTD, or focussing on a safety stock determination, have certain limitations in the available choice of some parameters and in that that they are largely deterministic, use specific probability density functions, or employ identical elements of echelon structure (Beamon, 1998; Axsäter, 2003). Further, the network configuration is usually confined to a two-echelon supply chain, with significantly fewer studies of other configurations such as spanning trees (Graves and Willems, 2000; Simchi-Levi and Zhao, 2005). Many models specifically employ Poisson process (or its variations) for external demand and Normal distribution for the lead-time demand (e.g. Ettl et al., 2000). A specific probability density function, though quite valid in certain cases, significantly narrows the problem and leaves aside many real-world demand patterns (Rossetti and Ünlü, 2011). The Poisson process, for example, is not always valid as was demonstrated by Smith and Dekker (1997). Also, as will be shown later some LTD distributions have little resemblance to Normal, supporting the conclusion made by Lau and Lau (2003) that for many situations and contrary to the prevalent view the often used Normal approximation for LTD is not robust. Many of the shortfalls of the existing multi-echelon inventory modelling have been summarised by Cattani et al. (2011) who also discuss the unjustified limitation on the network functionalities, e.g. a central warehouse supplies only its remote nodes and not the local consumers, which often is not the case. Another artificial limitation that is present in many multi-echelon models is the 'one node–one supplier' construct, whereas multiple supplies of identical items to a single store is quite common.

This paper extends generality of multi-echelon modelling and incorporates a fully flexible choice of parameters for a fairly general network configuration. The main idea is to propose an analytical framework for calculating a probability distribution of the demand occurring during lead-time, i.e. LTD distribution, for a single product for any node in a supply chain organised in an arbitrary stochastic network. The supply chain multi-echelon network can be organised in various graphic configurations. In this paper we focus on the polytree configuration, which is defined as a graph with at most one undirected path between any two nodes, i.e. a network of nodes connected by directed arrows (arcs) in such a way that no loops of any kind (directed or undirected) are allowed. The polytree configuration is sufficiently general (only the disallowance of loops separates it from a fully arbitrary network), so we have chosen it as the base configuration and only this configuration is thoroughly investigated. There are examples in the literature of a network with loops but these are mainly production/assembly chains that do not explicitly stipulate external supply/demand for intermediate nodes (e.g. Shu and Karimi, 2009). The polytree configuration is well suitable for modelling production/assembly networks too, though it best describes the acquisition and distribution of identical supplier-independent items, e.g. fuel, commodities, or food.

The intention was not only to develop a general and flexible model suitable for a wide variety of network implementations but to allow model reducibility. This means that if a network lacks certain stochastic elements or lacks certain vertices or edges it may still be solved with the presented formalism where the relevant formulae are reduced accordingly.

Requirements of the model were that both the demand and the lead-time be stochastic in general. In addition, the concept of intermittent (sporadic) demand or the uncertainty of the demand occurring period was introduced into the model. This describes a distribution of the periods when the demand does occur with the periods when it does not. In this model we equate the terms period and unit time that can be a day, a week, a month, etc. The intermittent demand often occurs for many items in industrial as well as military sectors. It was first studied by Croston (1972) and its modelling and forecast is showed by many researchers, e.g. Segerstedt (1994), Syntetos and Boylan (2010), to be an important consideration. Together with the demand quantity distribution the inter-demand time distribution creates a compound two dimensional distribution for the demand. Combined with a randomly distributed lead-time we form a three-way stochastic process.

The inventory policy of the model is a known (s, nQ) batch policy for discrete time and continuous demand quantity. Each node of a network represents a warehouse, tank or any inventory storage facility and arcs represent product movements. Our intention is not to focus on the strict mathematical derivation, but to emphasise how the formulation may be used to obtain the LTD distribution for any node in any polytree supply chain. The