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Efficient estimation of lower and upper bounds for pricing higher-dimensional American arithmetic average options by approximating their payoff functions☆

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ABSTRACT

In this paper, we develop an efficient payoff function approximation approach to estimating lower and upper bounds for pricing American arithmetic average options with a large number of underlying assets. The crucial step in the approach is to find a geometric mean which is more tractable than and highly correlated with a given arithmetic mean. Then the optimal exercise strategy for the resultant American geometric average option is used to obtain a low-biased estimator for the corresponding American arithmetic average option. This method is particularly efficient for asset prices modeled by jump-diffusion processes with deterministic volatilities because the geometric mean is always a one-dimensional Markov process regardless of the number of underlying assets and thus is free from the curse of dimensionality. Another appealing feature of our method is that it provides an extremely efficient way to obtain tight upper bounds with no nested simulation involved as opposed to some existing duality approaches. Various numerical examples with up to 50 underlying stocks suggest that our algorithm is able to produce computationally efficient results.

1. Introduction

The importance of American-style options has been growing increasingly and pricing of American options especially high-dimensional cases remains one of the challenging problems both theoretically and practically in the option pricing theory. In particular, high-dimensional American options would be valuable research topics. For example, Shiu, Chou, & Sheu (2013) document that basket warrants, essentially basket options with multiple underlying assets become more popular over the past decade.[2] In this paper, we focus on pricing American arithmetic average options. The appealing advantage of an American arithmetic average option lies in the fact that it exactly replicates the evolution of the portfolio formed by the underlying assets. For example, the cost of hedging a portfolio with an American arithmetic average option is much lower than a portfolio of individual options on the same underlying assets since the former takes the correlations among the underlying assets into account and only one option is involved in hedging. Besides, it would be simple for investors to replicate the payoff of any portfolio without actually holding the portfolio if there is such an American arithmetic average option available on the market.

Given these significant applications, efficient pricing methods for American arithmetic average options written on the average of multiple underlying assets are of great value from various points of view such as hedging and risk management especially after the recent financial crisis that re-emphasized the importance of risk management. The purpose of this paper is to develop an efficient approach to obtaining lower and upper bounds for American arithmetic average option prices on a large number of underlying assets.

The traditional valuation methods, such as lattice and tree-based techniques, for pricing high dimensional American option pricing problems are typically plagued by the curse of dimensionality and thus, simulation-based numerical methods are inevitably required. Earlier literature about simulation-based approaches can be traced back to Boyle (1977) in which European style claim is priced with Monte Carlo (MC) simulation. American style option pricing techniques with MC simulation include Bundling Methods in Tilley (1993), Stratified State Aggregation (SSA) in Barraquand & Martineau (1995), Stochastic Mesh Method (SMM) in Broadie & Glasserman (2004), regression-based approach in Tsitsiklis & Van Roy (1999) and Longstaff & Schwartz (2001), among others.

The existing simulation-based methods can be categorized into: (1) Primal approach, which aims to obtain a lower bound for an American option by estimating a suboptimal exercise strategy, e.g., regression-based approaches as in Tsitsiklis & Van Roy (1999) and Longstaff & Schwartz (2001); (2) Duality approach, which estimates an upper bound for an American option by using a dual martingale, e.g. Rogers (2002); Haugh & Kogan (2004) and Andersen & Broadie (2004).

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[1] Tel.: +44 07547269657.
[2] Although they focus on dealing with European basket warrants, essentially basket options, the American-style ones would be more important in practice.

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Among existing primal approaches, the most important simulation-based method is the regression-based approach, where computational costs are approximately linear in exercise opportunities and the number of simulated paths. The theory has been well established in Carriere (1996); Tsitsiklis & Van Roy (1999) and Longstaff & Schwartz (2001), etc. Related convergence analysis and simulation issues can be found in Tsitsiklis & Van Roy (2001); Clément, Lamberton, & Protter (2002); Glasserman & Yu (2004a, 2004b) and Stentoft (2004).

In particular, the least squares method (LSM) developed by Longstaff & Schwartz (2001) is the most widely used method due to its simplicity and generality. A lower bound of an American option can be obtained from a suboptimal optimal exercise strategy derived from linear regression procedure. However, this method and other primal approaches are becoming computationally expensive with the increasing dimension of pricing problem and hence the trade-off between computational costs and efficiency of approximation would be a critical issue.

A variety of methods have been proposed to improve the performance of regression-based approaches. For instance, to address arbitrary style of continuation values, Kohler, Krzyzak, & Todorovic (2010) use least square neural network regression estimates and estimate continuation values from artificial MC simulated paths. Their approach is more general than LSM since the regression is nonparametric. But, compared to LSM, the nonparametric in Kohler et al. (2010) would be even worse to implement for pricing high-dimensional American options.[3] More recently, Jain & Oosterlee (2012) proposed a stochastic grid method (SGM) which could be regarded as a hybrid of Barraquand & Martineau (1995): stratified sampling along payoff method, Longstaff & Schwartz (2001): Least square Monte Carlo method & Broadie & Glasserman (2004): stochastic mesh method. The proposed SGM algorithm is more suitable for pricing some high-dimensional American options than existing methods. However, SGM would be computationally costly when sub-simulations are embedded and more early exercise times are allowed.

To circumvent the curse of the dimensionality problem associated with pricing of multi-dimensional American options, several dimension reduction methods have been proposed. For example, Barraquand & Martineau (1995) introduce a partitioning algorithm. Their method differs from Tilley's bundling algorithm in that they partition the payoff space instead of the state space. Hence, only a one-dimensional space is partitioned at each time step, regardless the dimension of the problem. More recently, Jin, Li, Tan, & Wu (2013) further integrate this idea into state-space partitioning algorithm (SSPM) developed by Jin, Tan, & Sun (2007) and improve the computational efficiency significantly with computational accuracy preserved. Those papers, however, do not provide an algorithm for upper bounds.

In the present paper, we follow the dimension reduction approach to pricing high-dimensional American arithmetic average options. The key idea is to find a highly correlated geometric average for a given arithmetic average. As will become clear later, the former is more tractable than the latter in the sense that the geometric average has a lower dimension[4] than the corresponding arithmetic average, and thus the optimal exercise strategy for the American geometric average option is far easier to obtain than for the American arithmetic average option. In particular, when the asset prices are modeled by jump-diffusion processes with deterministic volatilities, the geometric mean is always a one-dimensional Markov process regardless of the number of underlying assets, and thus is free from the curse of dimensionality. Then the optimal exercise strategy for the American geometric average option is used to obtain a lower bound for the corresponding American arithmetic average option. In addition, by using an inequality similar to (4) in Haugh & Kogan (2004), we provide an extremely fast way to obtain the corresponding upper bound without nested MC simulations. To be more specific, in the inequality (4) in Haugh & Kogan (2004), we approximate the payoff function of given American arithmetic average option by the one of a highly correlated American geometric average option. Unlike Haugh & Kogan (2004), we do not need to find the optimal supermartingale and thus we do not need nested MC simulations.

An important limitation of the lower bound is that it is not easy to evaluate the accuracy of its approximation to the true option price. Upper bounds in combination with the corresponding lower bounds allow us to measure the accuracy of price estimators for American average options. In earlier literature, Broadie & Glasserman (1997, 2004) propose stochastic mesh methods which generate not only lower but also upper bounds and both bounds converge asymptotically to the true value. Despite the advantage of obtaining the upper bound, the stochastic mesh methods are quite computationally demanding. Boyle, Kolkiewicz, & Tan (2003) further generalizes Broadie & Glasserman (1997, 2004) with a low-discrepancy sequence for efficiency.

Independently developed by Rogers (2002); Andersen & Broadie (2004) and Haugh & Kogan (2004), duality approach is the most general technique among those upper bound related approach. The idea is to introduce a dual martingale in the pricing problem and rewrite the primal problem into a dual minimization problem. For example, Andersen & Broadie (2004) use nested MC simulation to approximate the optimal exercise strategy. On the other hand, Haugh & Kogan (2004) apply an intensive neural network algorithm and low discrepancy sequences to estimate the option prices. However, their estimation techniques to estimate dual martingale do not preserve the martingale property in general and the computational cost is generally high.

To improve this, Glasserman & Yu (2004b) proposed a special regression algorithm to preserve the martingale property. Nonetheless, the martingale property Condition (C3) on the basis functions may not be straightforward to verify in practice. In terms of efficiency, Kolodko & Schoenmakers (2004) try to overcome the computational inefficiency of nested simulation by choosing a different estimator to reduce the number of inner path simulations. However, the upper bound is not guaranteed by their estimator as the number of inner path is too few.

Instead of estimating a dual martingale directly, Belomestny, Bender, & Schoenmakers (2009) estimate the coefficient of the corresponding martingale representation of the dual martingale. By martingale representation theorem, the martingale property of the estimated dual martingale is preserved. The resultant bound is then the true upper bound. More recently, Zhu, Ye, & Zhou (2014) extend the method in Belomestny et al. (2009) to a jump-diffusion model. Their theoretical analysis shows that the martingale property of the estimated optimal dual martingale is preserved and no nested simulation is used in their algorithm. These methods, however, may become impractical for pricing high-dimensional American options as a regression-based method similar to LSM is employed to estimate dual martingales. By contrast, our upper bound algorithm requires neither nested simulation nor high-dimensional regression especially when the asset prices are modeled by exponential jump-diffusion processes with deterministic volatilities.

In summary, we have made two contributions to the literature of pricing high-dimensional American arithmetic average options. First, we have developed a computationally efficient dimension reduction method to estimate lower bound. Second, we provide an easy-to-implement approach to evaluate the upper bound which involves no nested simulation and is based on a simple linear regression procedure. We are not aware of any research in the current literature that estimates lower and/or upper bounds for pricing high-dimensional American

[3] We thank an anonymous referee for pointing out this to us.

[4] For assets following GBM, the geometric average is always one-dimensional Markov process. However, the dimension will increase if other state variables are involved such as stochastic volatility. Consider a case where there are ten stocks and the price of each stock follows Heston stochastic volatility model. Then, an arithmetic average depends on twenty state variables, namely, ten stock price processes and ten volatility processes. By contrast, the corresponding geometric average depends on eleven state variables, that is, the geometric average process itself and the ten volatility processes.