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Linear programming-based estimators in nonnegative autoregression

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ABSTRACT

This note studies robust estimation of the autoregressive (AR) parameter in a nonlinear, nonnegative AR model driven by nonnegative errors. It is shown that a linear programming estimator (LPE), considered by Nielsen and Shephard (2003) among others, remains consistent under severe model misspecification. Consequently, the LPE can be used to test for, and seek sources of, misspecification when a pure autoregression cannot satisfactorily describe the data generating process, and to isolate certain trend, seasonal or cyclical components. Simple and quite general conditions under which the LPE is strongly consistent in the presence of serially dependent, non-identically distributed or otherwise misspecified errors are given, and a brief review of the literature on LP-based estimators in nonnegative autoregression is presented. Finite-sample properties of the LPE are investigated in an extensive simulation study covering a wide range of model misspecifications. A small scale empirical study, employing a volatility proxy to model and forecast latent daily return volatility of three major stock market indexes, illustrates the potential usefulness of the LPE.

1. Introduction

In the last decades, nonlinear and nonstationary time series analysis have gained much attention. This attention is mainly motivated by evidence that many real life time series are non-Gaussian with a structure that evolves over time. For example, many economic time series are known to show nonlinear features such as cycles, asymmetries, time irreversibility, jumps, thresholds, heteroskedasticity and combinations thereof. This note considers robust estimation in a (potentially) misspecified nonlinear, nonnegative autoregressive model, that may be a useful tool for describing the behaviour of a broad class of nonnegative time series.

For nonlinear time series models it is common to assume that the errors are i.i.d. with zero-mean and finite variance. Recently, however, there has been considerable interest in nonnegative models. See, e.g., Abraham and Balakrishna (1999); Engle (2002); Tsai and Chan (2006); Lanne (2006) and Shephard and Sheppard (2010). The motivation to consider such models comes from the need to account for the nonnegative nature of certain time series. Examples from finance include variables such as absolute or squared returns, bid-ask spreads, trade volumes, trade durations, and standard volatility proxies such as realized variance, realized bipower variation (Barndorff-Nielsen and Shephard, 2004) or realized kernel (Barndorff-Nielsen et al., 2008).[1] This note considers a

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[1] Another example is temperature, which can be used for pricing weather derivatives (e.g. Campbell and Diebold, 2005; Alexandridis and Zapranis, 2013).

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nonlinear, nonnegative autoregressive model driven by nonnegative errors. More specifically, it considers robust estimation of the AR parameter β in the autoregression

$$y_t = \beta f(y_{t-1}, \dots, y_{t-s}) + u_t, \quad (1)$$

with nonnegative (possibly) misspecified errors u_t. Potential distributions for u_t include lognormal, gamma, uniform, Weibull, inverse Gaussian, Pareto and mixtures of them. In some applications, robust estimation of the AR parameter is of interest in its own right. One example is point forecasting, as described in Preve et al. (2015). Another is seeking sources of model misspecification. In recognition of this fact, this note focuses explicitly on the robust estimation of β in (1). If the function f is known, a natural estimator for β given the sample $y_1, \dots, y_n$ of size n and the nonnegativity of the errors is

$$\hat{\beta}_n = \min\left\{\frac{y_{s+1}}{f(y_s, \dots, y_1)}, \dots, \frac{y_n}{f(y_{n-1}, \dots, y_{n-s})}\right\}. \quad (2)$$

This estimator has been used to estimate β in certain restricted first-order autoregressive, AR(1), models (e.g. Anděl, 1989b; Datta and McCormick, 1995; Nielsen and Shephard, 2003). An early reference of the autoregression in (1) is Bell and Smith (1986), who considers the linear AR(1) specification $f(y_{t-1}, \dots, y_{t-s}) = y_{t-1}$ to model water pollution and the accompanying estimator in (2) for estimation.[2] The estimator in (2) can, under some additional conditions, be viewed as the solution to the linear programming problem of maximizing the objective function $g(\beta) = \beta$ subject to the $n-s$ linear constraints $y_t - \beta f(y_{t-1}, \dots, y_{t-s}) \geqslant 0$ (cf. Feigin and Resnick, 1994). Because of this, we will refer to it as a LP-based estimator or LPE. As it happens, (2) is also the (on $y_1, \dots, y_s$) conditional maximum likelihood estimator (MLE) for β when the errors are exponentially distributed (cf. Anděl, 1989a). What is interesting, however, is that $\hat{\beta}_n$ is a strongly consistent estimator of β for a wide range of error distributions, thus the LPE is *also* a quasi-MLE (QMLE).

In all of the above references the errors are assumed to be i.i.d. To the authors knowledge, there has so far been no attempt to investigate the statistical properties of LP-based estimators in a *non* i.i.d. time series setting. This is the focus of the present note. In that sense, the note can be viewed as a companion note to Preve and Medeiros (2011) in which the authors establish statistical properties of a LPE in a non i.i.d. *cross-sectional* setting. Estimation of time series models with dependent, non-identically distributed errors is important for two reasons: First, the assumption of independent, identically distributed errors is a serious restriction. In practice, possible causes for non i.i.d. or misspecified errors include omitted variables, measurement errors and regime changes. Second, traditional estimators, like the least squares estimator, may be inconsistent when the errors are misspecified. In some applications the errors may also be heavy-tailed. The main theoretical contribution of the note is to provide conditions under which the LPE in (2) is consistent for the unknown AR parameter in (1) when the errors are serially dependent, non-identically distributed and heavy-tailed.

The remainder of this note is organized as follows. In Section 2 we give simple and quite general conditions under which the LPE is a strongly consistent estimator for the AR parameter, relaxing the assumption of i.i.d. errors significantly. In doing so, we also briefly review the literature on LP-based estimators in nonnegative autoregression. Section 3 reports the simulation results of an extensive Monte Carlo study investigating the finite-sample performance of the LPE and at the same time illustrating its robustness to various types of model misspecification. Section 4 reports the results of a small scale empirical study, and Section 5 concludes. Mathematical proofs are collected in the Appendix. An extended Appendix (EA) available on request from the author contains some results mentioned in the text but omitted from the note to save space.

[2] Bell and Smith (1986) refer to the LPE as a 'quick and dirty' nonparametric point estimator.

2. Theoretical results

In finance, many time series models can be written in the form $y_t = \sum_{i=1}^{p} \beta_i f_i(y_{t-1}, \dots, y_{t-s}) + u_t$. A recent example is Corsi's (2009) HAR model.[3] In this section we focus on the particular case when $p = 1$ and the errors are nonnegative, serially correlated, possibly heterogeneously distributed and heavy-tailed random variables. The case when $p = 1$ is special in our setting as the linear programming problem of maximizing the objective function $g(\beta_1, \dots, \beta_p) = \sum_{i=1}^{p} \beta_i$ subject to the $n-s$ linear constraints

$$y_t - \sum_{i=1}^{p} \beta_i f_i(y_{t-1}, \dots, y_{t-s}) \geqslant 0$$

(cf. Feigin and Resnick, 1994) then has an explicit solution. This simplifies the statistical analysis of the LPE. In general ($p > 1$), one has to rely on numerical methods.

2.1. Assumptions

We give simple and quite general assumptions under which the LPE converges with probability one or almost surely (a.s.) to the unknown AR parameter.

Assumption 1. The autoregression $\{y_t\}$ is given by

$$y_t = \beta f(y_{t-1}, \dots, y_{t-s}) + u_t, \quad t = s+1, s+2, \dots$$

for some function $f : \mathbb{R}^s \to \mathbb{R}$, AR parameter $\beta > 0$, and (a.s.) positive initial values $y_1, \dots, y_s$. The errors u_t driving the process are nonnegative random variables.

Assumption 1 includes error distributions supported on $[\eta, \infty)$, for any unknown nonnegative constant η, indicating that an intercept in the process is superfluous (Section 3.1.2). It also allows us to consider various mixture distributions that can account for data characteristics such as jumps (Section 3.3.2). The next assumption concerns the potentially multi-variable function f, which allows for various lagged or seasonal specifications (Section 3.1.3).

Assumption 2. The function $f : \mathbb{R}^s \to \mathbb{R}$ is known (measurable and nonstochastic), and there exist constants $c > 0$ and $r \in \{1, \dots, s\}$ such that $f(\mathbf{x}) = f(x_1, \dots, x_r, \dots, x_s) \geqslant c x_r$ when all of its arguments are nonnegative.

Assumptions 1 and 2 combined ensure the nonnegativity of $\{y_t\}$, indicating that the process may be used to model durations, volatility proxies, and so on. Assumption 2 is, for instance, met by elementary one-variable functions such as e^{x_s}, $\sinh x_s$ and any polynomial in x_s of degree higher than 0 with positive coefficients.[4] Thus, in contrast to Anděl (1989b), we allow f to be non-monotonic.

Assumption 3. The error at time t is given by

$$u_t = \mu_t + \sigma_t \varepsilon_t, \quad t = s+1, s+2, \dots$$

[3] The HAR model of Corsi can be written as $y_t = \sum_{i=1}^{3} \beta_i f_i(y_{t-1}, \dots, y_{t-22}) + u_t$, where $f_1(y_{t-1}, \dots, y_{t-22}) = y_{t-1}$, $f_2(y_{t-1}, \dots, y_{t-22}) = y_{t-2} + \dots + y_{t-5}$, $f_3(y_{t-1}, \dots, y_{t-22}) = y_{t-6} + \dots + y_{t-22}$ and y_t is the realized volatility over day t. Here $p = 3$ and $s = 22$.

[4] An interesting example of a multi-variable function f is given by the AR index process considered by Im et al. (2006) for which $f(\mathbf{x}) = x_1 + \dots + x_s$ or, equivalently, $f(y_{t-1}, \dots, y_{t-s}) = y_{t-1} + \dots + y_{t-s}$. The AR index models of order 1, 5 and 22 all can be viewed as special cases of Corsi's (2009) HAR model.