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Detection of arbitrage in a market with multi-asset derivatives and known risk-neutral marginals

Bertrand Tavin *

EMLYON Business School, 23 Avenue Guy de Collongue, 69130 Ecully, France

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ABSTRACT

In this paper we study the existence of arbitrage opportunities in a multi-asset market when risk-neutral marginal distributions of asset prices are known. We first propose an intuitive characterization of the absence of arbitrage opportunities in terms of copula functions. We then address the problem of detecting the presence of arbitrage by formalizing its resolution in two distinct ways that are both suitable for the use of optimization algorithms. The first method is valid in the general multivariate case and is based on Bernstein copulas that are dense in the set of all copula functions. The second one is easier to work with but is only valid in the bivariate case. It relies on results about improved Fréchet–Hoeffding bounds in presence of additional information. For both methods, details of implementation steps and empirical applications are provided.

1. Introduction

The notion of arbitrage is fundamental in economics and finance, as it underpins the setup in which academics and practitioners solve issues about equilibrium, portfolio allocation and contingent claim valuation. In these disciplines, many theoretical developments are thus built on the absence of arbitrage opportunity as a central assumption. For institutions involved in the financial industry, it is a strategic issue to ensure that their systems do not produce such opportunities. Hence, the availability of methods to detect arbitrage is of the utmost interest.

In a market with a single underlying asset and a given set of vanilla options, the assessment of the absence of arbitrage is addressed in Carr and Madan (2005), Davis and Hobson (2007) and Cousot (2007). Essentially, the set of option prices is free of arbitrage as soon as butterfly spreads, call spreads and calendar spreads have positive prices.

Assessing the absence of arbitrage among a set of derivative prices becomes a much more involved task when the set under scrutiny has some exotic options in addition to vanillas, or in the case of a market with multiple underlying assets. The concern of our paper is to address the latter case in a general way that does not rely on the structure of a particular payoff and that is valid beyond the two-dimensional case. To the best of our knowledge, it has not yet been done in the existing literature. Our setup corresponds to a one period multi-asset market with known risk-neutral marginals, in which we obtain a characterization of the absence of arbitrage among a set of derivative prices in terms of copula functions. This characterization allows us to derive two necessary conditions of no-arbitrage, one of which is also sufficient, that both naturally lead to detection methods in the sense that if a condition is not verified then the market is not free of arbitrage. Hence our contribution is twofold. First, from a theoretical standpoint it allows a better understanding of the absence of arbitrage in our market model. Second, with practical perspectives, we detail the detection methods that are deduced from the theoretical part and we apply them to real market situations.

For a single risky asset, it is possible to build risk-neutral diffusions that are compatible with a given set of vanilla options. Early references on that topic are Dupire (1993) and Laurent and Leisen (2000). The former considers a local volatility diffusion coefficient and the latter considers the construction of a risk-neutral Markov chain consistent with observed call option prices. It is possible to

* Tel.: +33 (0)478337800; fax: +33 (0)478336169.
E-mail address: tavin@em-lyon.com

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go further and to obtain no-arbitrage bounds for an additional derivative when some are already available. This question has already been partially addressed. It is closely related, yet different from our concern. An approach to obtain the desired bounds for Asian options is based on the concept of comonotonicity, see Chen et al. (2008) and references therein. Another approach to obtain the desired bounds is via a Skorokhod embedding problem formulation, see Hobson (2010) and references therein. Yet another, more recent, approach is to apply optimal mass transportation theory to obtain the desired bounds, see Beiglbock et al. (2011) and Galichon et al. (2014).

In the multi-asset case, bounds on prices of options written on several underlyings are available. When marginals are known, upper and lower bounds for two-asset basket options are initially obtained in Dhaene and Goovaerts (1996), in a context of actuarial analysis of portfolios of dependent risks. The same upper and lower bounds are obtained for other two-asset option payoffs in Rapuch and Roncalli (2001). Tankov (2011) derives improved bounds when some two-asset options are already quoted. For basket options, when single underlying vanillas are quoted, upper and lower bounds are available and the associated replicating strategies are explicit. The lower bound result is only valid in the two-asset case. As for Asian options written on a single asset, the above mentioned comonotonicity approach can be used, see Dhaene et al. (2002a,b) and Vanmaele et al. (2006). Key results for basket options are in Hobson et al. (2005a) and Chen et al. (2008). See also Laurence and Wang (2005) and Hobson et al. (2005b). In d'Aspremont and Ghaoui (2006), the authors work with a linear programming approach and obtain upper and lower bounds on basket option price when other basket options, with different weights, are already available. In Deelstra et al. (2008) the case of Asian basket options is studied in a constant volatility Black–Scholes–Merton framework, these options are path-dependent multi-asset options. Upper and lower bounds are also available for spread options. The respective bounds are obtained in Laurence and Wang (2008, 2009). The case of spread options is particular because arbitrage opportunities did exist during year 2009 among such options written on Constant Maturity Swap rates. This occurrence is documented in McCloud (2011).

The pricing of European options written on several underlying assets has been widely studied. This body of research is linked to our problem but, as it is, does not answer it. The classical approach is to postulate a joint distribution for the underlying asset price returns and to calibrate the distribution parameters to available data in order to obtain prices and hedge ratios. For example, with this approach (Margrabe, 1978 and Stulz, 1982) both work in a two-asset extension of the Black–Scholes-Merton model and obtain valuation formulas, respectively, for exchange and rainbow options (also called min–max options). Alexander and Scourse (2004) propose a bivariate distribution built as a mixture for the pricing and hedging of spread options. Dempster et al. (2008) also study spread options and directly model the spread process in a cointegrated two-commodity framework. Nevertheless, in many cases, it is preferable to proceed in two steps by first specifying the marginals and then choosing the dependence structure. This alternative approach relies on the power of copula functions for the modeling of dependence and it allows an easier identification and understanding of the potential sources of risk. See among others (Rapuch and Roncalli, 2001; Coutant et al., 2001; Cherubini and Luciano, 2002 and Rosenberg, 2003).

The remainder of the paper is organized as follows. In Section 2 we explain our financial framework and we propose a characterization of the absence of arbitrage in terms of copula functions. In Section 3 we develop a first methodology based on the family of Bernstein copulas. In Section 4 we propose, for the two-asset case, another methodology based on improved Fréchet–Hoeffding bounds. Section 5 concludes.

2. Arbitrage and copulas in a multi-asset market

We begin this section by detailing the structure of our market model and formalizing our problem. We then introduce copula functions and deduce a twofold characterization of the absence of arbitrage in our market in terms of such functions.

2.1. Model and assumptions

We consider a fundamental probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{P}$ the historical probability measure. Our financial market has one period and $n+1$ non-redundant primary assets ($n \geqslant 2$), $t = 0$ is the initial time and $t = T < +\infty$ is the final time. The primary assets are denoted by $\left(B, S^1, \ldots, S^n\right)$. Their initial prices $\left(B_0, S_0^1, \ldots, S_0^n\right) \in]0, +\infty[^{n+1}$ are known (non-random) and their final prices are positive random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ and are denoted by $\left(B_T, S_T^1, \ldots, S_T^n\right)$. The 0^{th} asset, B, is a risk-free asset. It earns the risk-free rate $r \geqslant 0$ and its final value is non-random and known at initial time, $B_T = 1$. We suppose vanilla call options of all positive strikes to be available for the n risky assets of our market. For $i = 1, \ldots, n$, we denote $C^i(K^i)$ the call option written on S^i and struck at $K^i \in [0, +\infty[$ with the special case $C^i(0) = S^i$. Its final payoff is written $C_T^i(K^i) = \left(S_T^i - K^i\right)^+$ and $C_0^i(K^i)$ denotes its initial price.

Our financial market model departs from reality on two notable characteristics. First, we consider a one-period market ($T = 1$) where, in reality, trading can almost be done in continuous time. This assumption corresponds to a restriction of trading strategies to only static strategies. By static strategies we mean buy at initial time and hold until final time. Second, we assume the availability of vanilla call prices for a continuum of positive strikes. In reality, vanilla options are traded only at a finite number of strikes hence leaving space for ambiguity in empirical applications. This remaining ambiguity is well documented and can be kept acceptable for underlyings with liquidly traded options such as equity indices or foreign exchange rates. See among others (Jackwerth and Rubinstein, 1996).

We now introduce the notion of Risk-Neutral Measure for our financial market. The set of such measures is the cornerstone of the results presented in this paper because it is linked to the existence of arbitrage.

Definition 1 (*Risk-Neutral Measure*). A probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$, equivalent to $\mathbb{P}$, is a Risk-Neutral Measure (RNM) if, for $i = 1, \ldots, n$

$$C_0^i(K^i) = B_0 \mathbb{E}^{\mathbb{Q}}\left[C_T^i(K^i)\right] \text{ for all } K^i \in [0, +\infty[\quad (1)$$

We define $\mathcal{Q}$ as the set of Risk-Neutral Measures for our basic financial market.

The First Fundamental Theorem of Asset Pricing establishes the link between the set of risk-neutral measures and the absence of arbitrage opportunity. It has been first obtained in discrete time in Harrisson and Kreps (1979) and in continuous time in Harrisson and Pliska (1981). This theorem states that there is no arbitrage opportunity in the financial market if and only if $\mathcal{Q}$ is non-empty. For proofs, details, further references and extensions see Föllmer and Schied (2002) and Delbaen and Schachermayer (2006).