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# Almost marginal conditional stochastic dominance

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ABSTRACT

Marginal Conditional Stochastic Dominance (MCSD) developed by Shalit and Yitzhaki (1994) gives the conditions under which all risk-averse individuals prefer to increase the share of one risky asset over another in a given portfolio. In this paper, we extend this concept to provide conditions under which most (and not all) risk-averse investors behave in this way. Instead of stochastic dominance rules, almost stochastic dominance is used to assess the superiority of one asset over another in a given portfolio. Switching from MCSD to Almost MCSD (AMCSD) helps to reconcile common practices in asset allocation and the decision rules supporting stochastic dominance relations. A financial application is further provided to demonstrate that using AMCSD can indeed improve investment efficiency.



## 1. Introduction

The most common investment rule is certainly the mean–variance (MV) rule. It is easy to compute, and in some cases even to express analytically, which explains why the MV rule has become most widely accepted throughout the financial profession (see Lizyayev and Ruszczyński, 2012). On the other hand, Expected utility (EU) maximization lies at the heart of modern investment theory and practice. To be analytically consistent with EU maximization, the MV rule requires strong assumptions (such as quadratic utility functions or normally distributed returns), which seldom hold in practice. However, EU requires the specification of the investor's utility function which appears extremely difficult.

Stochastic dominance (SD) is an alternative approach which avoids all these shortcomings by considering the preferences shared by all the rational decision-makers. Therefore, it does not require a specific utility function nor a specific return distribution. Furthermore, it uses the whole probability distribution rather than the usual MV parameters of standard deviation and mean return. The second-degree stochastic dominance (SSD) rule is appropriate for the class of all risk-averse EU maximizers. It has the advantage that it requires no restrictions on probability distributions nor on investors' utility functions outside of the requirement that investors be risk-averse, EU maximizers.

Given a portfolio of assets, marginal conditional stochastic dominance (MCSD) has been introduced by Yitzhaki and Olkin (1991) and Shalit and Yitzhaki (1994) as a condition under which all risk-averse EU maximizer individuals prefer to increase the share of one risky asset over that of another. Specifically, these authors consider risk-averse investors holding a given portfolio of risky assets and derive criteria expressed in terms of the joint probability distribution of the assets and of the underlying portfolio to ensure that the share of an asset is increased at the expense of another in the portfolio. This helps to detect inefficiency and to improve inefficient portfolios. MCSD has been successfully applied to solve asset allocation problems by several authors, including Clark et al. (2011), Clark and Kassimatis (2012, 2013), Shalit and Yitzhaki (2010). MCSD expresses the conditions under which all risk-averse investors holding a specific portfolio prefer one asset to another. Furthermore, MCSD has been shown to involve more than pairwise comparisons as developed by Shalit and Yitzhaki (2003). It is a less demanding concept and more adapted to empirical analysis than SSD because it considers only marginal changes of holding risky assets in a given portfolio.

Despite their theoretical attractiveness, MV and SSD rules may create paradoxes in the sense that they fail to distinguish between some risky prospects, whereas it is obvious that the vast majority of investors would prefer one over the other. This is why Bali et al. (2009) considered almost stochastic dominance (ASD) as a viable

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alternative. ASD corresponds to all utility functions after eliminating pathological preferences, keeping only the economically relevant utility functions. Bali et al. (2009) demonstrated that the ASD rule unambiguously supports some common practice, like advising a higher stock to bond ratio for long investment horizons. Switching from SD to ASD thus allows for the provision of a theoretical support for the practitioners' view within the EU paradigm. The study conducted by these authors suggests that modifying MCSD into almost MCSD (AMCSD) may also help in the analysis of economic behavior under risk. This is the subject of the present work.

In this paper, MCSD is weakened to ensure that most (but not all) risk-averse decision-makers increase the share of one risky asset over another. This extension of MCSD to AMCSD is inspired by almost stochastic dominance rules introduced by Leshno and Levy (2002), and suitably corrected by Tzeng et al. (2013).[2] Specifically, restrictions are imposed on the marginal utility function and on its derivative to exclude extreme forms of preferences that are not shared by real-world investors. Then, the condition leading to MCSD is adapted to correspond to utilities defining almost second-degree stochastic dominance. As pointed out by Levy et al. (2010), investment rules based on stochastic dominance may cover "theoretical preferences that are not encountered in practice": there are situations where stochastic dominance is unable to rank two portfolios, whereas experimentally 100% of the subjects reveal a clear-cut ranking. The switch from MCSD to AMCSD can be expected to avoid such paradoxical results.

The remainder of this paper is organized as follows. The next section extends MCSD to AMCSD. Section 3 discusses a numerical example comparing the two concepts. Section 4 allows for changes in multiple assets in MCSD rules. Section 5 provides empirical illustrations. Section 6 briefly concludes the paper and discusses how to extend AMCSD rules to higher orders.

## 2. Almost marginal conditional stochastic dominance

### 2.1. Marginal conditional stochastic dominance

Assume that a risk-averse investor with a utility function $u$ holds a portfolio with $n$ risky assets. Let $w_0$ be the initial wealth, $X_i$ denote the rate of return on risky asset $i$ and $\alpha_i$ be the investment proportion on asset $i, i = 1, 2, \ldots, n$. A portfolio $\boldsymbol{\alpha}$ is defined by the shares $\alpha_i$ such that $\sum_{i=1}^{n}\alpha_i = 1$. The final wealth of the investor is given by $W = w_0(1 + \sum_{i=1}^{n}\alpha_i X_i)$. Henceforth, we normalize the initial wealth $w_0$ to unity so that $W = 1 + \sum_{i=1}^{n}\alpha_i X_i$.

The goal of the investor is to select the weights to maximize $E[u(W)]$. Given a portfolio $\boldsymbol{\alpha}$, Shalit and Yitzhaki (1994) have established that it is optimal to increase the weight $\alpha_k$ of asset $k$ at the expense of asset $j$ if, and only if,

$$E\left[u'(W)(X_k - X_j)\right] \geqslant 0. \tag{1}$$

Asset $k$ dominates asset $j$ according to MCSD if condition (1) is fulfilled for all risk-averse investors, that is, for all concave utility $u$.

Let $R$ denote the portfolio return, i.e.,

$$R = \sum_{i=1}^{n}\alpha_i X_i.$$

Shalit and Yitzhaki (1994) proved that for a given portfolio $\boldsymbol{\alpha}$, asset $k$ dominates asset $j$ according to MCSD if, and only if, the inequality

$$E[X_k|R \leqslant r] \geqslant E[X_j|R \leqslant r]$$

holds for all the return levels $r$. This is easily deduced from (1) by taking the kinked utilities $u(x) = \min\{x, r\}$. In words, MCSD favors assets performing better in adverse situations (i.e., when the portfolio underperforms $\iff R \leqslant r$).

The next section shows how to define AMCSD as distinct from MCSD, avoiding extreme forms of preferences.

### 2.2. From MCSD to AMCSD

MCSD is based on all the non-decreasing and concave utility functions, that is, on the utility functions in

$$U_2 = \{\text{utility functions } u|u' \geqslant 0 \text{ and } u'' \leqslant 0\}.$$

As explained in Leshno and Levy (2002), $U_2$ contains some extreme utility functions which presumably rarely represent real-world investors' preferences. The prototype is $u(x) = \min\{x, r\}$ for some constant $r$. Note that such utilities form the representative set of non-decreasing and concave utility functions used by Hadar and Seo (1988).

To reveal a preference for most investors, but not for all of them, we restrict $U_2$ to a subset of it. Specifically, following Leshno and Levy (2002), let us further impose restrictions on the utility function and define

$$U_2^*(\varepsilon) = \left\{u \in U_2| - u''(x) \leqslant \inf\{-u''(x)\}\left(\frac{1}{\varepsilon} - 1\right) \text{ for all } x\right\}, \tag{2}$$

where $\varepsilon \in \left(0, \frac{1}{2}\right)$. The range of the parameter $\varepsilon$ which controls the area of violation has been discussed empirically by Levy et al. (2010).

The following result characterizes the situations where asset $j$ is dominated by asset $k$ for all investors with $u \in U_2^*(\varepsilon)$. Before stating it formally, we need to introduce some additional notation. Let $\mu_i(r)$ denote the conditional expected return of asset $i$ when the portfolio return is $r$, i.e.,

$$\mu_i(r) = E[X_i|R = r].$$

Henceforth, we assume without real loss of generality that the return is bounded and valued over some interval $[a, b]$ of the real line. Furthermore, define

$$\begin{aligned} B(t) = &\int_a^t \left(\mu_k(r) - \mu_j(r)\right)dF_R(r) \\ = &\left(E[X_k|R \leqslant t] - E[X_j|R \leqslant t]\right)F_R(t) \\ \Omega = &\{t \in [a, b]|B(t) < 0\} \end{aligned}$$

and let $\Omega^c$ denote the complement of $\Omega$ in $[a, b]$. MCSD requires $B(t) \geqslant 0$ for all $t$, that is, $\Omega = \emptyset$. If this is not the case, $\Omega$ represents the set of violations for MCSD.

**Proposition 1.** *Given portfolio $\boldsymbol{\alpha}$, asset $k$ dominates asset $j$ for all individuals with preferences represented by the utility function $u \in U_2^*(\varepsilon)$ if, and only if,*

$$\int_{\Omega}(-B(t)dt) \leqslant \varepsilon \int_a^b |B(t)| \, dt \tag{3}$$

*and $E[X_k] \geqslant E[X_j]$.*

The proof of this result can be found in the appendix. Together with the comparison of expected returns, condition (3) provides the operational way to check for AMCSD in a given portfolio.

Tzeng et al. (2013) have shown that a distribution is preferred to another one by all decision makers with utility function $u \in U_2^*(\varepsilon)$ if and only if the distribution dominates the other one in terms of almost second-degree stochastic dominance, which contains two conditions. The first one is that the mean of the distribution is greater than that of the other one, which corresponds

[2] Lizyayev and Ruszczyński (2012) provided an alternative definition of almost stochastic dominance called tractable almost stochastic dominance due to its benefits in regard to tractability in computation.