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Pricing discrete path-dependent options under a double exponential jump–diffusion model

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ABSTRACT

We provide methodologies to price discretely monitored exotic options when the underlying evolves according to a double exponential jump diffusion process. We show that discrete barrier or lookback options can be approximately priced by their continuous counterparts' pricing formulae with a simple continuity correction. The correction is justified theoretically via extending the corrected diffusion method of Siegmund (1985). We also discuss the jump effects on the performance of this continuity correction method. Numerical results show that this continuity correction performs very well especially when the proportion of jump volatility to total volatility is small. Therefore, our method is sufficiently of use for most of time.

1. Introduction

Discrete (path-dependent) options are financial derivatives whose payoffs are determined by the underlying asset price levels at some preset points of time. This feature is also called "discrete monitoring". Among the many types of discrete options, two of the most popular ones are called *barrier options* and *lookback options*. Because of its popularity, knowing how to price a discrete barrier or lookback option quickly and accurately becomes a critical task in practice. Generally, such discrete scheme does not allow for a closed-form pricing formula, making experts resort to approximation methods instead.

One attempt to address this problem, initiated by Broadie et al. (1997), is to approximate a discrete option price by the price of a theoretical counterpart that assumes "continuous monitoring" (i.e., the payoff depends on the whole underlying price level during the contract life). This approach is termed as "continuity correction" in the literature. Related works include Broadie et al. (1999), Kou (2003), and Hörfelt (2003), among others. However, all of these studies regarding barrier and lookback options were conducted under the classical Black–Scholes (BS) Model, which is commonly recognized as unsuitable for describing the empirical distribution of a financial asset price.

Since their works, discretely monitored options have received more and more attention in the literature. There are many studies that have attempted to price such featured options under various models by proposing different numerical methods. For example, Ballestra et al. (2007) gave an approximation method for the transition probability to price exotic discrete options under Heston's stochastic volatility (SV) model, and Feng and Linetsky (2008) utilized Hilbert transforms to deal with discrete single- and double-barrier options in Lévy process-based models. Although these skills can approximate option prices well, typically they are not as efficient as the correction method.

Therefore, in this study, we follow the line of continuity correction approach but with a more sophisticated model. Specifically, we establish a such correction under Kou's (2002) Double Exponential Jump Diffusion Model (DEJDM), by proposing a modified Siegmund's corrected diffusion approximation. Surprisingly, the relevant correction terms do not seem to be affected by the presence of jumps, and hence remain the same as in the BS setting. One likely explanation is that the jump effect has been absorbed into the continuous-time pricing formula. Such offered analytical approximation formulas are of great use, not only for pricing, but also for risk management.

Although our technique seems to be applicable for general Jump Diffusion Models (JDMs), the verification involved is a bit too

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subtle and complicated to get the insight. In turn, we focus on DEJDM, which has analytical pricing formulae, and address some implementation issues on jump effects. Particularly, we notice that the proposed approximation is sufficiently useful, provided that the proportion of the total volatility coming from jump uncertainty is small. Therefore, our contribution to the literature covers not only theoretical justifications but also practical guidelines.

On the other hand, our approach of continuity correction (regarding barrier options) does not require restrictions on strike prices and barrier levels as they previously did in the BS studies. Numerical analysis also confirms this property. Since the chosen jump model nets the BS model, our results indeed provide a complete complement to the BS literature in this regard.

The rest of the paper will be developed as follows. Next section sketches the methodology and indicates the key concepts and difficulties behind, while Section 3 presents all resulting approximation formulae. A numerical analysis is given in Section 4, where we further explore the role of jumps. Final section concludes the study, and detailed proofs can be found in the Appendix.

2. Methodology overview

2.1. Barrier option

The payoff of a (European) barrier option on an asset S_t with maturity date $T > 0$ can be generally expressed as

$$\mathcal{X}_T = f(S_T) \times 1_{A_T}. \tag{1}$$

Expression (1) says that the holder of the option will be paid off an amount, $f(S_T)$, provided that the event A_T occurs during the contract life. Typically, the function f will be $(S_T - K)^+$ (call type) or $(K - S_T)^+$ (put type) with strike price K, while the event A_T describes a boundary-crossing problem that can be expressed as $\{\tau \leqslant T\}$ or $\{\tau > T\}$. Here, τ stands for the so-called first-passage time (stopping time), which can be an abbreviation of

$$\bar{\tau}(H) = \inf\{t \in \mathbb{R}_+ : S_t \geqslant H\} \quad (\underline{\tau}(H) = \inf\{t \in \mathbb{R}_+ : S_t \leqslant H\}) \tag{2}$$

or

$$\bar{\tau}_m(H) = \inf\{n \in \mathbb{Z}_+ : S_{n\cdot\Delta t} \geqslant H\} \quad (\underline{\tau}_m(H) = \inf\{n \in \mathbb{Z}_+ : S_{n\cdot\Delta t} \leqslant H\}). \tag{3}$$

In the above, H denotes the barrier level, and $\Delta t = T/m$ with $m \in \mathbb{N}$ being the frequency of monitoring. Note also that both $\mathbb{R}_+$ and $\mathbb{Z}_+$ include 0.

The path-dependent property of a barrier option is exactly captured by the notation τ. The τ described in parenthesis is viewed as of down-type (assuming $H < S_0$); otherwise, it belongs to up-type (assuming $H > S_0$ instead). According to the classical arbitrage-pricing theory, the price (V) of an option is just the expectation of its discounted payoff under a chosen pricing probability measure (P). That is to say, in the current case, the price of a barrier option at time zero is given by

$$V = E[e^{-rT}\mathcal{X}_T] \quad (= E_P[e^{-rT}\mathcal{X}_T], \quad \text{for notational convenience}). \tag{4}$$

Here, for simplicity, we assume the risk-free rate r is constant. Note that the market is incomplete in a JDM; thus, the measure P is just one risk-neutral measure. In this paper, we merely assume its existence[1] and all the settings and operations hereafter are directly based on this given P.

In this paper, options with feature (2) refer to continuous options, while those with (3) refer to discrete ones. Most option-pricing models assume (2) holds theoretically since Stochastic Calculus can be applied, leading to closed-form pricing formulas. However, in the real world, most contracts adopt a discrete scheme (3) for ease of implementation. Although the pricing theory is still applicable, no convenient formula can now be obtained.

Let $V(H)$ and $V_m(H)$ be, respectively, the initial value of a continuous barrier option and a discrete counterpart, with the other parameters being equal. At first, practitioners simply regarded these two values as approximately the same. This is true ideally especially when m is large, since the variables in (2) and (3) are said to converge weakly as Δt goes to zero. But it was later recognized that the rate of convergence would be so slow that significant pricing errors could arise.

For that reason, Broadie et al. (1997) proposed a way, by applying methods in Sequential Analysis, to accelerate the speed of convergence between V_m and V under the classic BS model. Specifically, they claimed that

$$V_m(H) = V(He^{\pm\sigma\sqrt{\Delta t}\beta}) + o(1/\sqrt{m}) \tag{5}$$

with some restrictions on K and H. Here $\pm$ is for case $\bar{\tau}$ (up-type)/$\underline{\tau}$ (down-type). Basically, the approximation formula (5) indicates that, one should shift away the barrier level first before directly applying the continuous-time formula to approximate the price value of a discrete counterpart. This approach is termed as "continuity correction" in the literature, and is now widely used in practice; see, for example, Chapter 25 of the textbook Hull (2011).

The intuition of such a correction is based on the following two observations under discrete monitoring: first, the boundary-crossing probability will be lower in discrete time than in continuous time; and second, there will always be an overshoot in discrete time. (We say an overshoot occurs if $S_\tau \neq H$, and denote by $|S_\tau - H|$ the amount of overshoot.) The overshoot phenomenon is shown in Fig. 1. Therefore, the shifting away of the barrier level accounts for the less boundary-crossing probability, and the adjusted (shifted) amount is somehow the expectation of the overshoot.

Note that there will be no overshoot with continuous monitoring under the BS model thanks to the continuous sample-path property of a diffusion process. Nevertheless, many empirical studies have shown that some financial asset processes have demonstrated discontinuities. Such a discontinuity is usually regarded as the result of a big event, like the bankruptcy of a company or a market crash. Straightforwardly, this makes researchers start to consider a JDM, the BS model plus a jump model, which can easily incorporate this discontinuity feature.

The situation for continuity correction becomes more complicated when we move to a JDM. The new challenge here will be at least twofold. First, there could be an overshoot even under con-

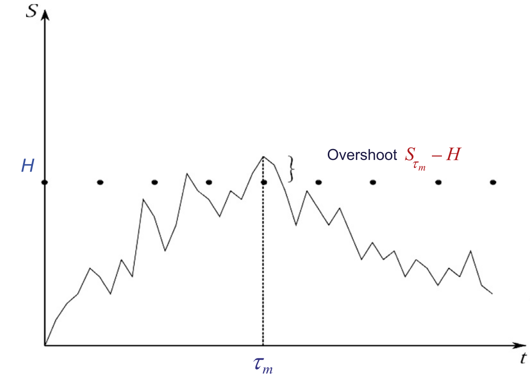

Fig. 1. An overshoot problem with discrete monitoring.

[1] For DEJDM, Kou (2002) showed a P can be determined by using a general equilibrium argument with a HARA-type utility for the representative agent.