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Static hedging and pricing American knock-in put options

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ABSTRACT

This paper extends the static hedging portfolio (SHP) approach of Derman et al. (1995) and Carr et al. (1998) to price and hedge American knock-in put options under the Black–Scholes model and the constant elasticity of variance (CEV) model. We use standard European calls (puts) to construct the SHPs for American up-and-in (down-and-in) puts. We also use theta-matching condition to improve the performance of the SHP approach. Numerical results indicate that the hedging effectiveness of a bi-monthly SHP is far less risky than that of a delta-hedging portfolio with daily rebalance. The numerical accuracy of the proposed method is comparable to the trinomial tree methods of Ritchken (1995) and Boyle and Tian (1999). Furthermore, the recalculation time (the term is explained in Section 1) of the option prices is much easier and quicker than the tree method when the stock price and/or time to maturity are changed.

1. Introduction

Barrier options are frequently used to reduce the hedging cost and are among the most popular options in the over-the-counter market. Thus the pricing and hedging of American barrier options are important (yet difficult) problems in the literature. In this paper, we extend the static hedging portfolio (SHP) approach of Derman et al. (1995) and Carr et al. (1998) to price and hedge American knock-in put options under the Black–Scholes model and the constant elasticity of variance (CEV) model of Cox (1975).[1] We first derive the American knock-in option values on the barrier and then construct a static hedging portfolio (SHP) which matches the knock-in option prices before maturity at evenly-spaced time points on the barrier. We use standard European calls to construct SHPs for American up-and-in put options and standard European puts to construct SHPs for American down-and-in put options, respectively.[2] In addition, we further apply the theta-matching condition to form the SHP such that the replication mismatches on the knock-in boundary can be significantly reduced and the performance of SHP can be improved.[3]

Because in-out parity does not hold for American barrier options, American knock-in option value is not equal to the difference between the corresponding plain vanilla American option price and the corresponding American knock-out option price. Therefore, although many numerical methods and analytical approximation formulae have been proposed to price American knock-out options, they cannot be applied to price American knock-in options even when the corresponding plain vanilla American option price is given.[4] To the best of our knowledge, only a few methods have been proposed for pricing American knock-in options. For example, AitSahlia et al. (2004) propose a numerical method to price American knock-in options. Haug (2001) presents analytic valuation formulae of American knock-in options but only for the case of $H \leqslant X$, where H and X are the barrier and strike prices, respectively. Dai and Kwok (2004) further extend Haug (2001) by deriving analytic valuation formulae for American knock-in options under all

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[1] The static hedging approach has been applied to hedge European barrier options statically beyond the Black–Scholes model, e.g. see Andersen et al. (2002), Fink (2003), Nalholm and Poulsen (2006), and Takahashi and Yamazaki (2009). In addition, beyond the barrier options, the static hedging approach has also been applied to other types of options such as American options (Chung and Shih (2009)), European-style Asian options (Albrecher et al. (2005)), and European-style installment options (Davis et al. (2001)).

[2] We will explain clearly how to construct a SHP in Section 2.

[3] Theta-matching condition means that the derivative of the SHP price with respect to time equals the derivative of the knock-in option price with respect to time on the barrier at the hedging time points. We will explain how to deal with the theta-matching condition in detail in Section 3.

[4] Boyle and Lau (1994), Ritchken (1995), Cheuk and Vorst (1996), and Chung and Shih (2007) develop lattice methods and Gao et al. (2000), AitSahlia et al. (2003), and Chang et al. (2007) develop analytical approximation formulae for pricing American knock-out options.

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possible cases. However, their analytic formulae are restricted to the Black–Scholes model only. In addition, standard American option values are involved in their analytic formulae and therefore numerical procedures are needed to obtain the prices of American knock-in options.

Concerning the hedging effectiveness, we show that even a bimonthly SHP is far less risky than a DHP with daily rebalancing for hedging American knock-in put options. In addition, in comparison to the existing methods, the proposed method has at least four advantages in pricing American knock-in put options. First of all, even if the existing numerical or analytical methods can price American knock-in options efficiently under the Black–Scholes model, it may be difficult to extend them to other asset-price processes. In contrast, the proposed method is applicable for more general underlying asset-price processes such as the constant elasticity of variance (CEV) model of Cox (1975). Secondly, we can directly derive the hedge ratios, such as delta and theta, of American knock-in options at the same time when we form the SHP. Thirdly, the proposed SHP method is more efficient than the modified trinomial tree method of Ritchken (1995) for pricing American knock-in options under the Black–Scholes model and the modified trinomial tree method of Boyle and Tian (1999) for pricing American knock-in options under the CEV model, when n (the number of time steps) is large.[5] Finally, the recalculation of the option prices and hedge ratios under the proposed method is much quicker (e.g. in some cases, 100 times faster) than the tree method when the stock price and/or time to maturity are changed. For instance, if the initial stock price is $85 and moves to $86 in the next minute, the proposed method takes less than 5% of the initial computational time to recalculate the American knock-in option price because there is no need to solve the SHP again (According to our numerical experiments, solving the SHP usually takes more than 95% of the total initial computational time.). In contrast, the recalculation time of most (if not all) existing tree methods is the same as their initial computational time because the lattice has to be built again in order to compute the option price.

The rest of the paper is organized as follows. Section 2 explains how to formulate the SHP for American knock-in put options. We discuss the improvement of the SHP with the theta-matching condition in Section 3. Section 4 compares the hedging effectiveness of SHPs and DHPs and discusses the numerical efficiency of the proposed method for pricing American knock-in put options under the Black–Scholes model. Section 5 presents the extensions to the CEV model. Section 6 concludes the paper.

2. Formulation of the SHP for American knock-in put options

In the following, we will first demonstrate how to construct the SHPs for American up-and-in options and American down-and-in options, respectively, under the Black–Scholes model.[6] When $H < X$ (where H is the knock-in boundary and X is the strike price), the knock-in boundary may overlap with the early exercise region and the boundary conditions on the barrier need to be carefully adjusted. Therefore, in the following, we consider American knock-in put options with $H > X$ and $H < X$, respectively.

2.1. Formulation of the SHP for an American up-and-in put option

In this subsection, we first consider an American up-and-in put option with $H > X$. Denote the price of an American up-and-in put as $F^{AUIP}(S, X, \sigma, r, q, t, T)$, where S, X, σ, r, q, t, and T are the stock price, the strike price, the return volatility, the risk-free rate, the dividend yield, the current time, and the maturity date, respectively. Under the Black–Scholes model, it is well known that the price of the American up-and-in put option satisfies the following partial differential equation (PDE):

$$\frac{1}{2}\sigma^2 S^2 F_{SS}^{AUIP} + (r-q)SF_S^{AUIP} + F_t^{AUIP} = rF^{AUIP}, \quad \text{for } S < H \text{ and } t < T. \tag{1}$$

In addition, the barrier boundary conditions and terminal conditions are

$$F^{AUIP}(H, X, \sigma, r, q, t, T) = F^{AP}(H, X, \sigma, r, q, t, T), \quad \text{if } t < T, \tag{2}$$

$$F^{AUIP}(S_T, X, \sigma, r, q, T, T) = \max(X - S_T, 0), \quad \text{if } \mathop{Sup}_{u \leqslant T} S_u \geqslant H, \tag{3}$$

$$F^{AUIP}(S_T, X, \sigma, r, q, T, T) = 0, \quad \text{if } \mathop{Sup}_{u \leqslant T} S_u < H, \tag{4}$$

where $F^{AP}(S, X, \sigma, r, q, t, T)$ is the price of the standard American put option. Eq. (2) states that the American up-and-in put price equals the corresponding standard American put price whenever the stock price hits the barrier price. Eqs. (3) and (4) are the payoff functions at the maturity date when the option is and is not knocked in, respectively.

Without reliance on a numerical method, we first follow Chung and Shih (2009) to construct an SHP for the corresponding American put and then obtain American put values on the knock-in boundary, i.e. $F^{AP}(H, X, \sigma, r, q, t, T)$ for $t \leqslant T$. The next step is to create a static portfolio of standard European options whose values match the American knock-in option values along the knock-in boundary. Specifically, suppose that we want to form an SHP of the American up-and-in put option which matches the barrier boundary conditions before maturity at n evenly-spaced time points, i.e. $t_0 = 0, t_1, \ldots, t_{n-1} = T - \Delta t$, where $\Delta t = T/n$. To match the knock-in boundary value at time t_i $(i = 0, 1, \ldots, n-1)$ on the knock-in boundary, we add w_i units of standard European call options maturing at time t_{i+1} with a strike price equaling H (why European call options are used will be explained later) into the SHP. We then solve w_i by matching the portfolio value with the standard American put price obtained in the first step (this step is called *value-matching* condition hereafter).

Similar to the binomial option pricing models, we work backward to determine the number of the standard European options for the above n-point SHP. For example, at time t_{n-1}, the *value-matching* condition on the barrier implies that

$$F^{AP}(H, X, \sigma, r, q, t_{n-1}, T) = w_{n-1} C^E(H, H, \sigma, r, q, t_{n-1}, T), \tag{5}$$

where $C^E(\cdot)$ is the European call price, i.e.

$$C^E(S, X, \sigma, r, q, t, T) = Se^{-q(T-t)}N(d_1) - Xe^{-r(T-t)}N(d_2),$$

$$d_1 = \frac{\ln(S/X) + (r - q + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \quad \text{and } d_2 = d_1 - \sigma\sqrt{T-t}.$$

At time t_{n-2}, we add w_{n-2} units of European calls maturing at time t_{n-1} with a strike price equaling H into the SHP and w_{n-2} is obtained by solving the following *value-matching* condition:

$$F^{AP}(H, X, \sigma, r, q, t_{n-2}, T) = w_{n-1} C^E(H, H, \sigma, r, q, t_{n-2}, T) + w_{n-2} C^E(H, H, \sigma, r, q, t_{n-2}, t_{n-1}). \tag{6}$$

Note that the newly added options at time t_{n-2} should not affect the existing solution of the SHP at time t_{n-1} on or within the barrier. The reason why we add the European call option maturing at time

[5] We explain the modified trinomial tree method of Ritchken (1995) in details in Appendix. The trinomial tree method of Boyle and Tian (1999) for pricing options under the CEV model is modified accordingly to price knock-in options in the same manners as described in Appendix.

[6] For the other diffusion models, the procedures for forming the SHP are similar except that the partial differential equations and the pricing formulae of the European options are different.