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Nonlinear portfolio selection using approximate parametric Value-at-Risk

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ABSTRACT

As the skewed return distribution is a prominent feature in nonlinear portfolio selection problems which involve derivative assets with nonlinear payoff structures, Value-at-Risk (VaR) is particularly suitable to serve as a risk measure in nonlinear portfolio selection. Unfortunately, the nonlinear portfolio selection formulation using VaR risk measure is in general a computationally intractable optimization problem. We investigate in this paper nonlinear portfolio selection models using approximate parametric Value-at-Risk. More specifically, we use first-order and second-order approximations of VaR for constructing portfolio selection models, and show that the portfolio selection models based on Delta-only, Delta–Gamma-normal and worst-case Delta–Gamma VaR approximations can be reformulated as second-order cone programs, which are polynomially solvable using interior-point methods. Our simulation and empirical results suggest that the model using Delta–Gamma-normal VaR approximation performs the best in terms of a balance between approximation accuracy and computational efficiency.

1. Introduction

In his ground-breaking work, Markowitz (1952) employs the variance of the portfolio return as the risk measure and establishes accordingly the mean–variance (MV) portfolio selection model. As the variance penalizes returns both below and above the mean without discrimination, it is evident that using variance as the risk measure is not suitable for dealing with the downside risk of a portfolio with an asymmetrical return distribution. Researchers in the field have thus proposed various new risk measures and numerous extensions of the mean–variance model since the pioneering work of Markowitz (1952). In particular, Value-at-Risk (VaR) has attracted much attention in recent years due to its conceptual simplicity and its wide applications in risk management (see, e.g., Duffie and Pan, 1997; Linsmeier and Pearson, 2000; Jorion, 2007).

The VaR of a portfolio can be defined as the maximum potential loss in portfolio value over a given time horizon under a specific confidence level. Although VaR is intuitive and simple in its definition, VaR has several shortcomings when concerning its conceptual and computational aspects. First, VaR disregards the loss beyond VaR, which may result in some uncontrollable and hazardous loss. Second, VaR does not always satisfy the subadditivity and therefore is not a coherent risk measure (Artzner et al., 1999). Third, the VaR of a portfolio is, in general, nonconvex with respect to the portfolio decision variables. The problem of minimizing the VaR of portfolios may possess multiple local minimizers (Mausser and Rosen, 1999). Therefore, portfolio optimization based on a VaR risk measure is computationally difficult. As a remedy of these deficiencies, Rockafellar and Uryasev (2000, 2002) propose a coherent risk measure, Conditional VaR (CVaR), also known as the expected shortfall, and reformulate the corresponding portfolio selection model as a convex optimization problem. Kou et al. (forthcoming), however, argue that the requirement of subadditivity may lead to risk measures that are not robust with respect to the underlying models and data, thus being unsuitable for regulatory purposes. They also show that VaR possesses some properties superior to CVaR, thus still an attractive risk measure.

In summary, VaR still remains a popular and standard tool in risk management, despite of its numerous shortcomings. VaR is especially suitable for nonlinear portfolio selection where payoff structures are nonlinear and the skewed return distribution is a prominent feature, for example, portfolio selection problems involving derivative assets. One of the successful applications of VaR in financial industry is RiskMetrics, a risk management system first developed by J. P. Morgan (1996). Calculating VaR of a given

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portfolio has become a daily practice in many financial institutions. As there is no closed-form expression for calculating the VaR for general nonlinear portfolios, simulation or approximation methods have to be used to calculate the VaR of a given portfolio. Historical simulation using only historical data is a simple and direct way for estimating VaR. The main disadvantage of historical simulation is that a forecast solely based on the past may not predict the future well in many situations. On the other hand, Monte Carlo simulation provides good estimation to the actual VaR of a portfolio via generating a large number of samples to approximate the distribution of the portfolio return. The procedure of Monte Carlo simulation, however, is time-consuming if one wants to achieve a reasonable accuracy of the estimation. An alternative way of calculating VaR is to use parametric approximation which is based on estimation of the actual change of the portfolio value by its first-order or second-order approximation and representing VaR as a function of the distribution parameters derived from this approximation. J. P. Morgan (1996), Britten-Jones and Schaefer (1999), Mina and Ulmer (1999) and Castellacci and Siclari (2003) investigate Delta-only (first-order) and Delta–Gamma (second-order) approximations of VaR under the normality assumption for the distributions of the underlying factors.

We focus in this paper on how to use parametric VaR in optimal nonlinear portfolio selection. Portfolio selection models using VaR risk measure estimated by historical simulation or Monte Carlo simulation are in general non-convex and non-smooth optimization problems and therefore computationally intractable. Larsen et al. (2002) propose heuristic algorithms based on CVaR to successively approximate VaR minimization problem. Pang and Leyffer (2004) reformulate VaR minimization problem as a linear program with equilibrium constraints (LPEC) and develop a branch-and-bound algorithm to search for a global optimal solution. Gaivoronski and Pflug (2005) propose a smoothing method for VaR minimization problem to generate an approximate optimal portfolio. Benati and Rizzi (2007) reformulate the VaR minimization problem as a 0–1 mixed integer programming problem, which can be solved by branch-and-bound methods. According to the capital requirement suggested by the Basel Accords, Santos et al. (2012) establish a general optimization model to minimize the capital requirement with the control on the number of VaR violations over the previous trading days and propose a convex approximation to overcome the discontinuity and discreteness of the resulting model. Although the above mentioned approaches can be applied to general nonlinear portfolio selection, there are only a few studies in the literature on optimal nonlinear portfolio selection. Isakov and Morard (2001) and Liang et al. (2008) discuss the selection of optioned portfolio in a mean–variance framework. Alexander et al. (2006) investigate the selection of derivative-only portfolio by minimizing CVaR via Monte Carlo simulation.

The main contribution of this paper is to investigate the computational aspects of optimal nonlinear portfolio selection models using parametric VaR approximations. In particular, we investigate the use of Delta-only, Delta–Gamma-normal and worst-case Delta–Gamma VaR approximations in nonlinear portfolio selection models. We show that several of these models can be reformulated as second-order cone programming (SOCP) problems which are computationally tractable as they can be solved polynomially by interior-point methods (see, e.g., Alizadeh and Goldfarb, 2003). We report extensive computational results to compare the accuracy and effectiveness of different parametric VaR approximations with partial and full Monte Carlo simulations for optimal portfolios generated by different VaR minimization portfolio selection models. We also analyze the performance of these models with real data from financial markets of Hong Kong, United States and France. Our numerical simulation and empirical results suggest that the model with Delta–Gamma-normal approximation of VaR yields the best performance for nonlinear portfolio selection in terms of the computational efficiency and robustness of the optimal portfolio solution.

We organize our paper as follows. In Section 2, we introduce different parametric approximations of VaR for a given nonlinear portfolio. We then establish in Section 3 portfolio selection models using these VaR approximations. In Section 4, we carry out simulation analysis for the accuracy and effectiveness of optimal portfolios generated by different nonlinear portfolio selection models. After performing empirical analysis in Section 5, we give some concluding remarks in Section 6.

2. Parametric approximations of VaR

In this section, we review several types of parametric approximations of VaR for a given portfolio. After introducing the basic concept of VaR, we discuss the first-order approximation method which leads to the simple Delta-only approximation of VaR. We then introduce the Delta–Gamma approximations resulted from the second-order approximation of the portfolio value change. Finally, we briefly discuss the Cornish–Fisher Delta–Gamma approximation.

2.1. Definition of VaR

Suppose that there are n derivative assets in a financial market, where asset i has value V_i at time t_0. Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ denote a given portfolio, where x_i is the holding amount of derivative asset i. The change of portfolio value over time period $[t_0, t_0 + \Delta t]$ can be expressed as:

$$\Delta V(\boldsymbol{x}) = \sum_{i=1}^{n} x_i \Delta V_i,$$

where ΔV_i is the value change of derivative asset i over $[t_0, t_0 + \Delta t]$. The Value-at-Risk (VaR) of portfolio $\boldsymbol{x}$ at confidence level c $(0.5 < c < 1)$ is defined as the smallest number u such that $-\Delta V(\boldsymbol{x})$ exceeds u with a probability not greater than $1 - c$, i.e.,

$$\mathrm{VaR}_c^*(\boldsymbol{x}) = \min\{u | \mathbb{P}(u < -\Delta V(\boldsymbol{x})) \leqslant 1 - c\}. \tag{1}$$

If $\Delta V(\boldsymbol{x})$ follows a normal distribution, then $\mathrm{VaR}_c^*(\boldsymbol{x})$ can be easily calculated as the c-quantile of $-\Delta V(\boldsymbol{x})$.

Assume now that value V_i of derivative asset i at time t_0 depends on a set of underlying factors $\boldsymbol{f} = (f_1, f_2, \ldots f_m)$. We can thus express V_i as $V_i = V_i(\boldsymbol{f}, t_0)$. Assume also that the change of the factors, denoted by $\Delta \boldsymbol{f}$, over the time period $[t_0, t_0 + \Delta t]$ follows a multi-normal distribution, namely,

$$\Delta \boldsymbol{f} \sim \mathcal{N}(\boldsymbol{\mu_f}, \Sigma),$$

which is a commonly used assumption that facilitates theoretical analysis and practical applications. Under this assumption, if all the values of derivative assets are *linear* functions of the underlying factors $\boldsymbol{f}$, then the VaR of the portfolio $\boldsymbol{x}$ is the quantile of a normal distribution. In general cases, however, the derivative assets are not linearly dependent on the underlying factors. Approximation approaches have been popular in financial industry to calculate the VaR of a portfolio. The basic idea is to approximate the change of portfolio value by the first-order and/or second-order terms in its Taylor expansion. Consequently, we obtain the Delta-only approximation and Delta–Gamma approximations, respectively (see Britten-Jones and Schaefer, 1999; Mina and Ulmer, 1999; Castellacci and Siclari, 2003).