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Models of the yield curve and the curvature of the implied forward rate function

Peter J. Yallup *

Manchester Business School, University of Manchester, Booth Street West, Manchester, M15 6PB, United Kingdom

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ABSTRACT

We examine several alternative models of the UK gilt yield curve using daily data for the period 12 July 1996–10 February 2010. We select the best models according to two criteria: low out of sample errors in pricing bonds and low curvature of the implied forward rate curve function. We suggest additions to some of the models that significantly improve their performance. Some of the new models out perform those typically used by the central banks. In particular this paper suggests that the model used by the Canadian Central Bank which both outperforms other models and is particularly easy to estimate, is well suited to the UK gilt market.

1. Introduction

The association between interest rates and term to maturity, known as the term structure of interest rates (or alternatively the "zero coupon" yield curve) is a fundamental relationship for central banks and other market participants. The yield curve is used to identify rich and cheap securities and to price new issues to the bond market. In addition, the yield curve is used to assess the impact of economic (particularly monetary) policy on the economy through its effect on current interest rates and also through the implied forward rate curve.

In this paper we examine many alternative models of the yield curve and we try to evaluate these across all practical parameter combinations. We select the best models according to two criteria which are most relevant to the expediency of the resulting yield curves. The first criterion is that a superior yield curve model should have low out of sample errors in pricing bonds; the second criterion is that a superior yield curve model should imply a forward rate curve with low curvature. We apply the selection criteria consistently across the different yield curve models.

We use fixed coupon bond data from the UK Government bond (gilt) market which is a large and liquid market. On 27 May 2005 the UK Treasury issued its first "ultra long" fixed coupon gilt, abruptly extending the yield curve from about 35 years to 50 years in maturity. The data used in this paper spans the period 12 July 1996–10 February 2010, and thus allows a comparison of the situation both pre and post this structural change.

We look at a large number of interesting alternative models including some which have not previously been used with UK data. We achieve far better fit to the data (both in sample and out of sample) and we explicitly calculate a measure of the curvature of the forward rate curve. Some of the new models outperform those typically used by the central banks. In particular, this paper proposes that the Li et al. (2001) model used by the Canadian Central Bank which both outperforms other models and is particularly easy to estimate is suitable for estimating the UK gilt yield curve.

1.1. Yield curve models

There are two groups of yield curve models which are present in the literature. The first group of yield curve model (dynamic term structure model) is motivated by the need to price options on long term interest rates (e.g. options on bonds). To do this requires modelling not only the yield curve but also the volatility of those interest rates. In addition, if a model is to be useful in a market where prices are evolving, accuracy in pricing the whole term structure may be less important than the ability to accurately model volatility and to price options quickly. Examples of this group of dynamic term structure models include Cox et al. (1985) and Heath et al. (1992). Determining which of the many alternative dynamic term structure models best fits a given set of data is examined in Aït-Sahalia and Kimmel (2010) and Joslin et al. (2011).

* Tel.: +44 (0) 161 2756333.
E-mail address: peter.yallup@mbs.ac.uk

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The second group of yield curve model is motivated by the need to model the yield curve alone accurately. Here the users of the yield curve models will be central banks and other participants in government bond markets (see Bank for International Settlements, 2005). Such models are used directly to identify rich and cheap bonds and to price new issues. In addition, yield curve models are used for calculating statistics (e.g. implied zero coupon interest rates) that are published for the benefit of all market participants and provide the key inputs to the essential task of determining an appropriate monetary policy,[1] as they show the markets expectation of the future level of interest rates. Thus, much of the published research on these models is conducted by the central banks themselves. It is on this second group of yield curve models that this research is focused.

Yield curve models can be expressed in terms of either the zero coupon yield curve, $r(t)$, the discount function curve, $d(t)$ or the instantaneous forward rate curve, $f(t)$. These are related as follows:

$$d(t) = \exp(-r(t)\cdot t) = \exp\left[-\int_0^t f(s)ds\right] \tag{1}$$

There are four types of model which we will evaluate. These are Nelson and Siegel, polynomial, discount function and cubic spline type models.

2. Estimating yield curve models, practical considerations

2.1. Objective function

To estimate a yield curve model, we need a measure of how well it fits the data. Assume that $\underline{\mathbf{P}}$ is a column vector of gilt prices which we use to estimate a yield curve model. Let $\underline{\mathbf{P}}^*$ be a column vector of price estimates for the gilts based upon their cash flows, discounted by the * yield curve model. Let $\underline{\mathbf{W}}$ be a square "weighting" matrix of appropriate dimension. $\underline{\mathbf{W}}$ has diagonal element w_{ii} > = 0 and zero off diagonal elements. The objective is to:

$$\text{Minimise}: (\underline{\mathbf{P}} - \underline{\mathbf{P}}^*)^T \underline{\mathbf{W}} (\underline{\mathbf{P}} - \underline{\mathbf{P}}^*) \tag{2}$$

Typically the weights w_{ii} are the reciprocal of either the duration or the modified duration of each individual bond squared (e.g. Anderson and Sleath (2001) use the reciprocal of the square of the modified duration).

The objective of this study is to estimate the yield curve. Errors in long term rates of interest will have a greater impact on the pricing of long bonds than the same changes in short term rates will have on short term bonds. In order to select the appropriate weighting scheme we need to understand how changes in yields (or the yield curve) affect the prices of coupon bonds. The modified duration of a bond gives the sensitivity of the price of a bond to changes in yields, or indeed the sensitivity of the yield of a bond to changes in price.

Consider a 5 and 10 year zero coupon bond yielding 5% annually. The modified duration gives the percentage change in the value if yields rise by one basis point (0.01%), (provided that any change in yield is small). Assume that the annual yield rises by 0.1 basis points; we can calculate the new price for each bond and confirm that the modified duration does indeed give the correct value (see Table 1). If we calculate continuously compounded yields the relationship between the modified duration and the change in prices is also valid. In this case the 'new price' of the zero coupon bonds is slightly lower (as a 0.1 basis point move in continuously compounded yields is a slightly larger move than a 0.1 basis point move in annual yields). Note that the duration and the modified duration are the same if we are using continuously compounded yields.

Hence if we use $[100/(\text{Price} \times \text{Modified Duration})]^2$ as the weight in Eq. (2) this will give the desired result. If all bond prices are near 100 this would be equivalent to using $[1/(\text{Modified Duration})]^2$ as the weight. Using the modified weighting scheme allows us to properly handle both high and low (or even zero) coupon bonds.

2.2. Over-fitting yield curve models

One possible problem with estimating yield curve models is the possibility of over-fitting the data. To illustrate, suppose that the yield curve is linear, but observations of yields are measured with some random error. If we have ten observations of yields we could fit a ninth power polynomial to the data exactly. The close fit would suggest that this elaborate model is the best model. Indeed adding terms to any model cannot reduce the fit to the data, as we add terms the fit will typically improve. We need to consider other ways of determining the 'best' model besides those that are purely related to the statistical fit of a model "in sample".

2.3. Effective range of yield curve models

When a yield curve model is estimated, the range of the data used will determine the effective range of the estimated model. For example, if the model is estimated using 1–10 year maturities, it will probably give unreliable estimates of 15 year yields. Indeed some models are not well behaved outside the range of the estimation data used. This consideration will be relevant when the various models are tested.

3. Models of the yield curve

3.1. Nelson and Siegel models of the yield curve

The Nelson and Siegel group of yield curve models comprise the Nelson and Siegel (1987) model and the related models of Svensson (1994) and of Bliss (1986). Nelson and Siegel (1987) introduced a model of the yield curve that is widely used.[2] The model was later extended by Svensson (1994). The Svensson model is specified in terms of the instantaneous forward rate function:

$$f(t) = \beta_0 + \beta_1 \exp\left(\frac{-t}{m_1}\right) + \beta_2\left(\frac{t}{m_1}\right)\exp\left(\frac{-t}{m_1}\right) + \beta_3\left(\frac{t}{m_2}\right)\exp\left(\frac{-t}{m_2}\right) \tag{3}$$

The Nelson and Siegel (1987) model is the same as the above model with $\beta_3 = 0$. Svensson (1994) introduced two extra parameters to give a greater variety of shapes to the instantaneous forward rate and yield curve curves. We can transform the model to give a closed form expression for the yield curve:

$$r(t) = \beta_0 + \beta_1\left(\frac{1-\exp\left(\frac{-t}{m_1}\right)}{\frac{t}{m_1}}\right) + \sum_{i=1}^{2}\beta_{i+1}\left(\frac{1-\exp\left(\frac{-t}{m_i}\right)}{\frac{t}{m_i}} - \exp\left(\frac{-t}{m_i}\right)\right) \tag{4}$$

As $t \to 0$, $r(t) \to \beta_0 + \beta_1$ (also $f(t) \to \beta_0 + \beta_1$) and as $t \to \infty$, $r(t) \to \beta_0$ (also $f(t) \to \beta_0$). So in this model β_0 can be interpreted as a long

[1] A striking recent example is the Bank of England's implementation of a policy of "quantitative easing ". Between 11 March 2009 and 26 January 2010 almost £ 200 billion was used by the Bank of England to purchase gilts from the market.

[2] Diebold and Li (2006) and Christensen et al. (2011) have derived dynamic versions of the Nelson and Siegel (1987) model.