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Robust optimization of conditional value at risk and portfolio selection ☆

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Abstract

This paper deals with a portfolio selection model in which the methodologies of robust optimization are used for the minimization of the conditional value at risk of a portfolio of shares.

Conditional value at risk, being in essence the mean shortfall at a specified confidence level, is a coherent risk measure which can hold account of the so called "tail risk" and is therefore an efficient and synthetic risk measure, which can overcome the drawbacks of the most famous and largely used VaR.

An important feature of our approach consists in the use of techniques of robust optimization to deal with uncertainty, in place of stochastic programming as proposed by Rockafellar and Uryasev. Moreover we succeeded in obtaining a linear robust copy of the bi-criteria minimization model proposed by Rockafellar and Uryasev. We suggest different approaches for the generation of input data, with special attention to the estimation of expected returns.

The relevance of our methodology is illustrated by a portfolio selection experiment on the Italian market.

JEL classification: C0; C1; G0; N2

Keywords: Coherent risk measures; Conditional value at risk; Robust optimization

1. Introduction

The practical relevance of portfolio selection models has constantly increased, since their introduction in the financial literature, due to the structural transfer of big private capitals toward investments generally not required by non institutional operators.

As a consequence, the interest of private and institutional investors for techniques and tools aimed at a more efficient forecast of the dynamics of securities prices and to a rational management of investment capital, is hugely increased.

The last aspect is the heart of this contribute that essentially consists in the application of robust optimization to the minimization of the conditional value at risk (CVaR) as a way to obtain efficient portfolios.

Classical portfolio selection models, still largely used for their conceptual simplicity and utility in applications, are based on a bi-criteria optimization scheme in which the goal is to form a portfolio in which expected return is maximized, while some index of risk is minimized.

In the classical Markowitz model risk is measured by means of a dispersion measure, such as variance or

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standard deviation. More recently, starting from the observation that positive and negative deviations of the returns from their mean play a greatly asymmetric role in the investor's perception, financial practice and related theory showed increasing interest towards quantile based measures, such as value at risk (VaR).

Value at Risk, if studied in the framework of coherent risk measures, lacks subadditivity, and therefore convexity, in the case of general loss distributions (although it may be subadditive for special classes of them, e.g. for normal distributions). This drawback entails both inconsistency with the well accepted principle of diversification (diversification reduces risk) and greater problems from the point of view of numerical tractability.

To overcome these problems, recent literature on portfolio selection focused on coherent risk measures and in particular on conditional value at risk (CVaR). Various papers (see for instance Rockafellar and Uryasev, 2000; Rockafellar and Uryasev, 2002; Pflug, 2000; Gaivoronski and Pflug, 2001) considered coherent risk measures as the objective to be minimized.

Another weak point of classical selection models has been recently illustrated: the optimization process leads to solutions which are likely to depend heavily on the parameters perturbations. As data are often, for a number of reasons, only known approximatively, this dependence makes the theoretical and numerical results highly unreliable for practical purposes.

This feature has been initially dealt with through the methods of stochastic programming and, in the last few years, with the help of a methodology which was recently introduced in the optimization literature (see BenTal and Nemirovski, 1998; BenTal and Nemirovski; BenTal and Nemirovski, 2002; Goldfarb and Iyengar, 2001; Vladimirou, 2003) under the name of robust optimization. We follow here this approach.

This paper is structured as follows: In Section 2 we will quickly present coherent risk measures which were introduced in Artzner et al. (1997), Artzner et al. (1999) to obtain, as underlined by the Authorities, more efficient risk measures. Particular evidence is given to the conditional value at risk (Rockafellar and Uryasev, 2002) and to its reformulation to which we will refer for our linear models of CVaR minimization.

In Section 3 we will analyze how risk and uncertainty emerging from parameters variability can be handled, focusing the attention on some techniques of robust optimization.

Many researchers suggest using these techniques when the unknown parameters are known, within some confidence level, to belong to certain intervals or variation ranges. The arguments presented in Sections 2 and 3 will justify the implementation of the robust version of the bicriteria model of selection proposed by Rockafellar and Uryasev (2002); this will be the subject of Section 4.

Note that, in general, the robust reformulation needs not be linear even in situation in which the original problem is linear. More frequently the new problem can be cast in the form of a second order cone programming or as a semi-definite programming problem (Goldfarb and Iyengar, 2001; Goldfarb and Iyengar, 2003; ElGhaoui and Labret, 1997) thus asking for more demanding minimization techniques.

An important feature of our analysis, maybe the most important, is that we are able to avoid this problem and obtain a linear robust reformulation of our problem, hence allowing for a standard minimization procedure. This goal is reached through the use of Soyster's approach to robust optimization.

With the aim to operate some comparisons that could help to evaluate the advantages of the robust optimization model, we performed a gradual implementation of the Rockafellar–Uryasev model; we assume the CVaR obtained by an appropriate historical simulation as a reliable risk measure, and formulate in three different versions the values of the expected return of every share, defined as capital gain.

We start with an estimation given by a weighed average of historical returns; due to the unsatisfactory results of the first version, we use a random walk approach to obtain a second type of estimations and, finally, we construct a robust version of the model using each previous estimation as center point for its uncertainty set. Note that our choices for the estimation of the expected returns do not coincide with those studied by Schottle and Werner (2005) or described by Bertsimas and Brown (2006), Bertsimas et al. (2007).

The comparisons among these three approaches consists in an ex-post analysis on the results obtained by each of them, which proves how the strategies obtained by means of the robust approach have a definitely better performance.

2. Coherent risk measures

In these last years the interest of the authorities towards the effects of unexpected losses connected with extremal events affecting financial markets has greatly increased. As a consequence, the attention towards the risks undertaken by financial institutions increased accordingly.

Since the late '80s, the Basel Committee (Basel Council, 1996a; Basel Council, 1996b; Basel Council, 1996c) has stressed the importance of a better and widely accepted specification for the meaning of risk and suggested the introduction of a standard model for its quantification, based on mathematical and statistical concepts.

This is the background that explains the choice of value at risk (VaR) as a synthetic risk measure, which can express the market risk of a financial asset or of a portfolio.

Roughly speaking, VaR is the maximum potential loss that a financial asset can suffer with a certain probability, during a certain holding period.

Formally it is then an appropriate quantile of the probability distribution of the random variable "loss of the financial asset at time T".