Journal of Econometrics 190 (2016) 212–221

Contents lists available at ScienceDirect

Journal of Econometrics

journal homepage: www.elsevier.com/locate/jeconom

A weak instrument F-test in linear IV models with multiple endogenous variables[☆]

Eleanor Sanderson[a,b], Frank Windmeijer[a,b,c,*]

[a] Department of Economics, University of Bristol, UK
[b] CMPO and IEU, University of Bristol, UK
[c] CEMMAP/IFS, London, UK

ARTICLE INFO

Article history:
Available online 20 June 2015

JEL classification:
C12
C36

Keywords:
Weak instruments
Multiple endogenous variables
F-test

ABSTRACT

We consider testing for weak instruments in a model with multiple endogenous variables. Unlike Stock and Yogo (2005), who considered a weak instruments problem where the rank of the matrix of reduced form parameters is near zero, here we consider a weak instruments problem of a near rank reduction of one in the matrix of reduced form parameters. For example, in a two-variable model, we consider weak instrument asymptotics of the form $\pi_1 = \delta\pi_2 + c/\sqrt{n}$ where π_1 and π_2 are the parameters in the two reduced-form equations, c is a vector of constants and n is the sample size. We investigate the use of a conditional first-stage F-statistic along the lines of the proposal by Angrist and Pischke (2009) and show that, unless $\delta = 0$, the variance in the denominator of their F-statistic needs to be adjusted in order to get a correct asymptotic distribution when testing the hypothesis $H_0 : \pi_1 = \delta\pi_2$. We show that a corrected conditional F-statistic is equivalent to the Cragg and Donald (1993) minimum eigenvalue rank test statistic, and is informative about the maximum total relative bias of the 2SLS estimator and the Wald tests size distortions. When $\delta = 0$ in the two-variable model, or when there are more than two endogenous variables, further information over and above the Cragg–Donald statistic can be obtained about the nature of the weak instrument problem by computing the conditional first-stage F-statistics.

1. Introduction

Following the work of Staiger and Stock (1997) and Stock and Yogo (2005), testing for weak instruments is now commonplace. For a single endogenous variable model, the standard first-stage F-statistic can be used to test for weakness of instruments, where weakness is expressed in terms of the size of the bias of the IV estimator relative to that of the OLS estimator, or in terms of the magnitude of the size distortion of the Wald test for parameter hypotheses. Stock and Yogo (2005) tabulated critical values for the standard F-statistic that have been incorporated in software packages.

For multiple endogenous variables, inspection of the individual first-stage F-statistics is no longer sufficient. The Cragg and Donald (1993) statistic can be used to evaluate the overall strength of the instruments in this case, and Stock and Yogo (2005) have tabulated critical values of the minimum eigenvalue of the Cragg–Donald statistic for testing weakness of instruments. They derive the limiting distributions under weak instrument asymptotics where the reduced form parameters are local to zero in each reduced form equation, and obtain critical values that are conservative in the sense that they are rejecting the null of weak instruments too infrequently when the null is true.

In this paper, we are interested in analysing tests for weak instruments in a model with multiple endogenous variables in a setting where the reduced form parameters are not local to zero, but where the reduced form parameter matrix is local to a rank reduction of one. In this case, the values of the F-statistics in each of the first-stage equations can be high, but the identification of (some of) the model parameters is weak. We will focus initially on a model with two endogenous variables. The weak instrument asymptotics we consider are local to a rank reduction of one, of the form

$$\pi_1 = \delta\pi_2 + c/\sqrt{n},$$

[☆] This research was funded by the Economic and Social Research Council (RES-343-28-0001), the European Research Council (DEVHEALTH-269874) and the Medical Research Council (MC_UU_12013/9). Helpful comments were provided by Jörn-Steffen Pischke, Mark Schaffer, Peter Schmidt, Chris Skeels, Jonathan Temple and two anonymous referees.

* Corresponding author at: Department of Economics, University of Bristol, UK.
E-mail address: F.Windmeijer@bristol.ac.uk (F. Windmeijer).

http://dx.doi.org/10.1016/j.jeconom.2015.06.004
0304-4076/

where π_1 and π_2 are the parameters in the two reduced-form equations, c is a vector of constants and n is the sample size. We call these asymptotics LRR1 weak instrument asymptotics.

We will focus solely on the properties of the 2SLS estimator. We investigate the use of a conditional first-stage F-statistic along the lines of the proposal by Angrist and Pischke (2009) and show that the variance formula in the denominator of their F-statistic needs to be adjusted in order to get a correct asymptotic distribution when testing the null hypothesis, in the two-variable model, $H_0 : \pi_1 = \delta\pi_2$. We further show that the resulting new conditional F-statistic is equivalent to the Cragg–Donald minimum eigenvalue statistic. Using our weak instrument asymptotics we show that this conditional F-statistic cannot be used in the same way as the Stock and Yogo (2005) procedure for a single endogenous variable to assess the magnitude of the relative bias of the 2SLS estimator of an individual structural parameter. This is because the OLS bias expression contains additional terms such that the expression for the bias of the 2SLS estimator relative to that of the OLS estimator does not have the simple expression as in the one-variable case. However, the total relative bias can be bounded as can the size distortions of Wald tests on the structural parameters.

In a two-endogenous-variable model the conditional F-statistics for each reduced form are equivalent to each other and to the Cragg–Donald minimum eigenvalue statistic under our LRR1 weak instrument asymptotics. This holds unless $\delta = 0$, in which case the local rank reduction is due to the fact that π_1 is local to zero and the first-stage F-statistic for x_1 will be small and that for x_2 will be large. In this case, both the Angrist–Pischke F-statistic and our conditional F-statistic for x_1 can be assessed against the Stock–Yogo critical value, and the 2SLS estimator for the structural parameter on x_2 is consistent. Additional information can also be obtained from our conditional F-statistics when there are more than two endogenous variables, as they will identify which variables cause the near rank reduction. For example, if in a three variable model the near rank reduction is due to the reduced form parameters on two variables only, the conditional F-statistic for the third variable will remain large giving the researcher valuable information about the nature of the problem and directions for solving it. We also show that the 2SLS estimator for the structural parameter of the third variable is consistent in that case.

The paper is organised as follows. In Section 2 we introduce the linear model with one endogenous variable and summarise the Staiger and Stock (1997) and Stock and Yogo (2005) results for testing for weak instruments. Section 3 considers weak instrument test statistics for the linear model with two endogenous explanatory variables and introduces the new conditional F-tests. Section 4 considers the relative bias and Wald test size distortions for the 2SLS estimator under the LRR1 weak instrument asymptotics as outlined above and presents some Monte Carlo results for the two-variable model. Section 4 also shows the usefulness of the conditional F-test statistics in a model with more than two endogenous variables. Finally, Section 5 concludes.

2. Weak instrument asymptotics in one-variable model

In this section we follow the basic Staiger and Stock (1997) and Stock and Yogo (2005) setup. The developments of the weak instrument setup and concepts for the one-variable model play an important role when we expand the model to multiple endogenous variables in the next section. The simple model is

$$y = x\beta + u, \tag{1}$$

where y, x, and u are $n \times 1$ vectors, with n the number of observations. There is endogeneity, such that $E(u|x) \neq 0$. The reduced form for x is

$$x = Z\pi + v, \tag{2}$$

where Z is a $n \times k_z$ matrix of instruments and v is $n \times 1$. For individual u_i and v_i we assume,

$$\begin{pmatrix} u_i \\ v_i \end{pmatrix} \sim (0, \Sigma); \quad \Sigma = \begin{pmatrix} \sigma_u^2 & \sigma_{uv} \\ \sigma_{uv} & \sigma_v^2 \end{pmatrix}.$$

The 2SLS estimator is given by

$$\widehat{\beta}_{2SLS} = \frac{x'P_Z y}{x'P_Z x} = \beta_0 + \frac{x'P_Z u}{x'P_Z x},$$

where $P_Z = Z\left(Z'Z\right)^{-1}Z'$.

The concentration parameter is given by

$$CP = \frac{\pi'Z'Z\pi}{\sigma_v^2}$$

and is a measure of the strength of the instruments, see Rothenberg (1984). A small concentration parameter is associated with a bias of the 2SLS estimator and deviations from its asymptotic normal distribution.

A simple test whether the instruments are related to x is of course a Wald or F-test for the hypothesis $H_0 : \pi = 0$. The Wald test is given by

$$W_\pi = \frac{\widehat{\pi}'Z'Z\widehat{\pi}}{\widehat{\sigma}_v^2} = \frac{x'Z\left(Z'Z\right)^{-1}Z'x}{\widehat{\sigma}_v^2},$$

where $\widehat{\pi} = \left(Z'Z\right)^{-1}Z'x$ is the first-stage OLS estimator, and $\widehat{\sigma}_v^2 = x'M_Z x/n$, where $M_Z = I - P_Z$. Under the null, $W_\pi \xrightarrow{d} \chi^2_{k_z}$. The F-test is given by $F = W_\pi/k_z$. Note that we refrain from a degrees of freedom correction in the variance estimate. Also, note that the analyses here and further below extend to a model with additional exogenous regressors, as we can replace y, x and Z everywhere by their residuals from regressions on those exogenous regressors.

Staiger and Stock (1997) introduce weak instrument asymptotics as a local to zero alternative, $\pi = c/\sqrt{n}$, which ensures that the concentration parameter does not increase with the sample size

$$CP = \frac{\pi'Z'Z\pi}{\sigma_v^2} \xrightarrow{p} \frac{c'Q_{ZZ}c}{\sigma_v^2},$$

where $Q_{ZZ} = \operatorname{plim}\left(n^{-1}Z'Z\right)$.

Assuming that conditions are fulfilled, such that

$$\begin{pmatrix} \frac{1}{\sqrt{n}}Z'u \\ \frac{1}{\sqrt{n}}Z'v \end{pmatrix} \xrightarrow{d} \begin{pmatrix} \psi_{Zu} \\ \psi_{Zv} \end{pmatrix} \sim N\left(0, \Sigma \otimes Q_{ZZ}\right),$$

and $k_z \geq 3$ when assessing relative bias. Then under weak instrument asymptotics,

$$\widehat{\beta}_{2SLS} - \beta = \frac{x'Z\left(Z'Z\right)^{-1}Z'u}{x'Z\left(Z'Z\right)^{-1}Z'x} \xrightarrow{d} \frac{\sigma_u}{\sigma_v}\frac{(\lambda + z_v)'z_u}{(\lambda + z_v)'(\lambda + z_v)}$$

where

$$\lambda = \sigma_v^{-1}Q_{ZZ}^{1/2}c; \qquad z_v = \sigma_v^{-1}Q_{ZZ}^{-1/2}\psi_{Zv}; \qquad z_u = \sigma_u^{-1}Q_{ZZ}^{-1/2}\psi_{Zu}.$$

The bias of the OLS estimator is given by

$$\begin{aligned}\widehat{\beta}_{OLS} - \beta &= \frac{x'u}{x'x} = \frac{(Z\pi + v)'u}{(Z\pi + v)'(Z\pi + v)} \\ &= \frac{n^{-1}\left(n^{-1/2}c'Z'u + v'u\right)}{n^{-1}\left(n^{-1}c'Z'Zc + 2n^{-1/2}c'Z'v + v'v\right)} \\ &\xrightarrow{p} \frac{\operatorname{plim} n^{-1}v'u}{\operatorname{plim} n^{-1}v'v} = \frac{\sigma_{uv}}{\sigma_v^2} = \frac{\sigma_u}{\sigma_v}\rho,\end{aligned}$$

where $\rho = \frac{\sigma_{uv}}{\sigma_u\sigma_v}$.