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Estimating jump–diffusions using closed-form likelihood expansions

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ABSTRACT

The indispensable role of likelihood expansions in financial econometrics for continuous-time models has been established since the ground-breaking work of Aït-Sahalia (1999, 2002a, 2008). Jump–diffusions play an important role in modeling a variety of economic and financial variables. As a significant generalization of Li (2013), we propose a new closed-form expansion for transition density of Poisson-driven jump–diffusion models and its application in maximum-likelihood estimation based on discretely sampled data. Technically speaking, our expansion is obtained by perturbing paths of the underlying model; correction terms can be calculated explicitly using any symbolic software. Numerical examples and Monte Carlo evidence for illustrating the performance of density expansion and the resulting approximate MLE are provided in order to demonstrate the practical applicability of the method. Using the theoretical results in Hayashi and Ishikawa (2012), some convergence properties related to the density expansion and the approximate MLE method can be justified under some standard sufficient (but not necessary) conditions.

1. Introduction

Continuous-time jump–diffusion processes have been widely used in various fields of science and technology for providing approximations to real-world dynamics of random fluctuations involving both relatively mild diffusive evolutions and sudden discontinuities caused by significant shocks. In financial economics, jump–diffusion models were introduced in the seminal work of Merton (1976), in which asset price is modeled by a combination of the celebrated Black–Scholes–Merton model (see Black and Scholes (1973) and Merton (1973)) and a compound Poisson process. During the past few decades, they have become a natural choice for modeling financial variables.

The literature has witnessed an explosion of developments and applications of jump–diffusion models in asset pricing, risk management and portfolio consumption optimization. Various stochastic volatility models with jump were proposed and investigated in, e.g., Bates (1996), Bates (2000), Duffie et al. (2000), Pan (2002), Johannes et al. (2003), and Broadie et al. (2007). By enriching both diffusive and jump components as well as their interactions, the affine jump–diffusion models were formally proposed in Duffie et al. (2000), which facilitate asset pricing and econometric analysis owing to their analytical tractability. For pricing various exotic options in using analytical methods, the double exponential jump–diffusion model was proposed by Kou (2002). By employing the backward induction principle based on the Hamilton–Jacobi–Bellman equations, portfolio planning problems involving jump risk were considered in, e.g., Liu et al. (2003), Pan and Liu (2003), Aït-Sahalia et al. (2009), Aït-Sahalia and Hurd (2015), and Jin and Zhang (2012). By enriching specifications of jump intensity according to the idea of Hawkes processes (see, e.g., Hawkes (1971)), self-exciting and mutual-exciting jumps are considered in, e.g., Aït-Sahalia et al. (2015), Aït-Sahalia and Hurd (2015), Errais et al. (2010), and Giesecke et al. (2011).

Econometric analysis of jump–diffusion models leads to issues that are significantly different from those typically encountered in discrete-time series analysis, e.g., the estimation of models formulated in continuous-time using data sampled at discrete-time intervals. To conduct likelihood-based inferences in this practical setting, transition densities play an important role; see, e.g., related discussions in Aït-Sahalia (2002b, 2004) and the references therein. Maximum-likelihood estimation (MLE hereafter) for jump–diffusions usually encounters challenges arising from time-consuming computation of transition densities. Closed-form expressions for transition densities cannot be obtained even for some simple jump–diffusion models,

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e.g., a jump–diffusion mean-reverting Ornstein–Uhlenbeck model. To conduct MLE, one usually needs computationally intensive numerical methods, e.g., Monte Carlo simulation and characteristic-function-based inversion method. Even if characteristic functions of the transition distribution exist in closed-form (e.g., for the affine jump–diffusion models proposed by Duffie et al. (2000)), the Fourier inversion based density evaluation suffers from a large computational load for each parameter set searched in the numerical procedure of optimization. Given a typically large number of possible candidate parameter sets and a large number of observations in high-frequency financial datasets (see, e.g., the survey in Mykland and Zhang (2010)), this is computationally expensive (if not impractical) because of the repeated valuation of numerical Fourier inversions in the whole procedure for MLE.

Among various methods for approximating transition densities, closed-form expansions have become popular because of their fast computing time and numerical accuracy. In particular, as a result of the progressive development of modern computation technology, calculation of high-order expansions will become increasingly feasible, and thus renders arbitrary accuracy at least in principle. For diffusion models, a milestone is the ground-breaking invention of Hermite-polynomial-based density expansion and its application in MLE proposed in Aït-Sahalia (1999, 2002a, 2008), which motivated various substantial refinements and applications, see, e.g., Bakshi et al. (2006), Aït-Sahalia and Mykland (2004, 2003), Aït-Sahalia and Kimmel (2007, 2010), Egorov et al. (2003), Xiu (2014), Chang and Chen (2011), Dipietro (2001), Stramer et al. (2010), and Choi (2013, 2015a,b). Enlightened by this stream of literature, various density expansions for jump–diffusion models were proposed, see, e.g., Aït-Sahalia and Yu (2006) for the application of saddle point approximation, Yu (2007) obtained from solving for correction terms of an expansion from Kolmogorov's forward and backward equations, Schaumburg (2001) for expanding transition density of a Levy-driven model on a related functional space, Filipović et al. (2013) for a general approximation theory in weighted Hilbert spaces for random variables, Giesecke and Schwenkler (2011) for approximating point process filters, as well as Choi (2015a) for approximating transition density function of a multivariate time-inhomogeneous jump–diffusion process in a closed-form expression.

Complementing to the existing methods, we will propose a new closed-form expansion for transition density and apply it in approximate MLE for multivariate Poisson-driven jump–diffusion models. Our method can be viewed as a significant extension of the method for diffusion models proposed in Li (2013). Because of the fundamental challenge led by adding jumps, our expansion starts from a new method of parametrization, which can be regarded as a path perturbation and is different from the small-time setting employed in Li (2013) for diffusion models. With presence of jumps, the calculation of correction terms involves various explicit computations related to both the diffusive and jump components. Following similar discussions in Li (2013) (see pp. 1351–1352), our expansion can be regarded as a jump–diffusion analogy of the celebrated Edgeworth-type expansions; see, e.g., Chapter 2 in Hall (1995) and applications to martingales in Mykland (1992, 1993). However, in contrast to the traditional Edgeworth expansions, our expansion does not require the knowledge of generally implicit moments, cumulants or characteristic function of the underlying variable, and thus it is applicable to a wide range of jump–diffusion processes.

The theoretical foundation for validity of our expansion originates in the theory of Watanabe (1987) and Yoshida (1992) for analyzing generalized Wiener functionals, as well as its theoretical generalization in Hayashi and Ishikawa (2012) for analyzing generalized Wiener–Poisson functionals, which focus on an alternative class of expansions relying on the theory of large-deviations. The uniform convergence rate (with respect to various parameters) of our density expansion for a parameterized jump–diffusion model can be proved under some standard sufficient conditions on the drift and diffusion coefficients. This leads to convergence of the resulting approximate MLE to the true MLE; and thus, the approximate MLE inherits the asymptotic properties of the true MLE. Such theoretical results will be supported by numerical tests and Monte Carlo simulations for some representative examples.

The rest of this paper is organized as follows. In Section 2, we introduce the model with some technical assumptions. In Section 3, we propose the transition density expansion with closed-form correction terms of any arbitrary order. In Section 4, numerical performance of the density expansion and Monte Carlo evidence for the resulting approximate MLE are demonstrated through examples. In Section 5, we conclude the paper and outline some opportunities for future research. Technical details on explicit calculation of expansion terms are provided in Appendices A–D. In an online supplementary material, Li and Chen (2016), we document some examples of closed-form expansion formulas, proofs of the results in the appendices, detailed calculation regarding some alternative specifications of the jump-size distribution, some theoretical discussions on the validity of our density expansion and the resulting approximate MLE.

2. The model and basic setup

We focus on a Poisson-driven jump–diffusion model governed by the following stochastic differential equation (SDE hereafter):

$$dX(t) = \mu(X(t);\theta)dt + \sigma(X(t);\theta)dW(t) + dJ(t;\theta), X(0) = x_0 \tag{1}$$

where $X(t)$ is a d-dimensional random vector; $\{W(t)\}$ is a d-dimensional standard Brownian motion; $\mu(x;\theta) = (\mu_1(x;\theta), \mu_2(x;\theta), \ldots, \mu_d(x;\theta))^\top$ is a d-dimensional vector-valued function and $\sigma = (\sigma_{ij}(x;\theta))_{d\times d}$ is a $d \times d$ matrix-valued function with an unknown parameter θ belonging to a multidimensional open bounded set Θ. Here, $J(t;\theta)$ is a vector-valued jump process modeled by a compound Poisson process which can be specified as

$$J(t;\theta) \equiv (J_1(t;\theta), J_2(t;\theta), \ldots, J_d(t;\theta))^\top := \sum_{n=1}^{N(t)} Z_n \equiv \sum_{n=1}^{N(t)} \left(Z_{n,1}, Z_{n,2}, \ldots, Z_{n,d}\right)^\top,$$

where $\{N(t)\}$ is a Poisson process with a constant intensity λ. For different integers n, $Z_n = (Z_{n,1}, Z_{n,2}, \ldots, Z_{n,d})^\top$ are i.i.d. multivariate random variables. Assuming $\tau_1, \tau_2, \ldots,$ are the jump arrival times, the jump path can be expressed as a step function, i.e.,

$$J(t;\theta) = \sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\left(Z_{i,1}, Z_{i,2}, \ldots, Z_{i,d}\right)^\top\right) 1_{[\tau_n,\tau_{n+1}]}(t). \tag{2}$$

Let $E \subset \mathbb{R}^d$ denote the state space of X.

We note that various popular jump–diffusion-based asset pricing models (see, e.g., Merton (1976), Kou (2002), Bates (2000), Duffie et al. (2000), and Broadie et al. (2007)) take or can be easily transformed into the form of (1). This model relaxes the condition on linear drift and diffusion of the affine jump–diffusion model proposed in Duffie et al. (2000). By assuming the intensity of $\{N(t)\}$ to be a constant, the existence and uniqueness of the solution to model (1) can be guaranteed under some technical conditions, see, e.g., discussions in Yu (2007). Besides, this assumption is supported by various empirical evidences, see, e.g., Bates (2000), Andersen et al. (2002), and Chernov et al. (2003). In modeling typical financial variables using a multidimensional jump–diffusion model, the small sample problem is usually severe in the estimation of