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Double asymptotics for explosive continuous time models[☆]

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ABSTRACT

This paper establishes a double asymptotic theory for explosive continuous time Lévy-driven processes and the corresponding exact discrete time models. The double asymptotic theory assumes the sample size diverges because the sampling interval (h) shrinks to zero and the time span (N) diverges. Both the simultaneous and sequential double asymptotic distributions are derived. In contrast to the long-time-span asymptotics ($N \to \infty$ with fixed h) where no invariance principle applies, the double asymptotic distribution is derived without assuming Gaussian errors, so an invariance principle applies, as the asymptotic theory for the mildly explosive process developed by Phillips and Magdalinos (2007). Like the in-fill asymptotics ($h \to 0$ with fixed N) of Perron (1991), the double asymptotic distribution explicitly depends on the initial condition. The convergence rate of the double asymptotics partially bridges that of the long-time-span asymptotics and that of the in-fill asymptotics. Monte Carlo evidence shows that the double asymptotic distribution works well in practically realistic situations and better approximates the finite sample distribution than the asymptotic distribution that is independent of the initial condition. Empirical applications to real Nasdaq prices highlight the difference between the new theory and the theory without taking the initial condition into account.

1. Introduction

Following the recent global financial crisis, one of the worst financial crises in history, public policy makers and academic researchers alike have devoted much effort into finding the causes of this crisis. A widely believed cause is the birth and burst of the U.S. real estate bubble. Not surprisingly, the recent literature focuses on the econometric identification of bubbles; see, for example, Phillips et al. (2011), Phillips and Yu (2011), Homm and Breitung (2012), and Phillips et al. (2015a,b, 2014). A primary technique used in this literature relies on the asymptotic theory developed in Phillips and Magdalinos (2007, hereafter PM) for a mildly explosive discrete time model.[1] The asymptotic distribution of PM is empirically appealing as it does not rely on the assumption of Gaussian errors, unlike the conventional asymptotic theory for the standard explosive model developed in White (1958) and Anderson (1959). Explosive processes are used for bubble analysis because, according to the rational expectations theory, the presence of bubble implies the explosive sub-martingale property. In the discrete time autoregressive set-up, this property leads to an autoregressive root larger than unity; see Gurkaynak (2008) for a recent survey of the literature on bubbles. In an empirical study, based on a recursive method implemented in a discrete time model proposed in Phillips et al. (2011), Phillips and Yu (2011) analyzed the bubble episodes in various U.S. markets and documented the bubble migration mechanism during the financial crisis. It was found that the real estate bubble in the U.S. indeed predates the financial crisis.

However, it is well-known that the degree of deviations from unity is typically determined by data frequency in discrete time models. Consequently, the empirical results may be sensitive to the choice of data frequency. Another potential restriction in using the theory of PM is that the asymptotic distribution is independent

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[1] The error term in PM is assumed to be a sequence of independent and identically distributed (i.i.d.) random variables. Magdalinos (2012) extended the asymptotic results of PM to the case where the errors are serially dependent. The asymptotics for the case in which the variance of the errors is infinite were established by Aue and Horvath (2007).

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of the initial condition. The initial condition was assumed to be of a small order in PM, and therefore, the resulting asymptotic distribution may not provide an accurate approximation to the finite sample distribution when the initial value is large.

In this paper, we overcome the two aforementioned problems in the literature by developing a double asymptotic theory for an explosive continuous time Lévy-driven process. There are several important reasons leading us to focus on a continuous time Lévy-driven process. First, it is well-known that the persistence parameter in continuous time models does not depend on data frequency (Bergstrom, 1990). Therefore, inference about the explosive behavior in the continuous time framework is less sensitive to the choice of data frequency in empirical analysis. Second, the continuous time model provides a natural tool to accommodate an initial condition whose order is higher than that in PM. As a result, our asymptotic distribution explicitly depends on the initial condition. This feature is the same as in the in-fill asymptotics ($h \to 0$) developed in Phillips (1987a) and Perron (1991). Not surprisingly, we find that our asymptotic theory improves over that does not depend on the initial condition in the finite sample approximation. Third, the use of Lévy-driven process allows us to develop an invariance principle for the persistence parameter, thereby sharing the asymptotic property of PM. The invariance principle is desirable because in many empirical analyses of bubbles the assumption of Gaussian errors may not be realistic.

The results in our paper build upon and extend the important work of Perron (1991). Based on the in-fill asymptotic scheme, Perron established a connection between the continuous diffusion process and the local-to-unity process and derived an alternative approximation to the estimator of the autoregressive parameter. His asymptotic theory permits explicit consideration of the effects of different initial conditions.[2] In our paper, by letting $h \to 0$ and $N \to \infty$ simultaneously, we build a link between the continuous time model and the discrete time autoregressive model with root moderately deviated from unity. Like Perron (1991), our double asymptotic distribution for the explosive case explicitly depends on the initial condition. However, when the process is stationary, as expected, no role for the initial condition is found in the double asymptotics.

Instead of focusing on the Brownian-motion-driven diffusion process as in Perron (1991), we consider the continuous time models driven by the Lévy process. Not only is our model empirically more realistic, but it also allows for the establishment of an invariance principle. Moreover, we derive two types of sequential asymptotics ($N \to \infty$ followed by $h \to 0$, and $h \to 0$ followed by $N \to \infty$) to bridge the gap among the simultaneous double asymptotics, the long-time-span asymptotics, and the in-fill asymptotics.

The results in our paper also extend the seminal work of PM (2007). In the explosive case, our double asymptotic scheme leads to a mildly explosive autoregressive model that has an initial condition with a higher order of magnitude than that in PM. That is why our asymptotic distribution depends on the initial condition, unlike the asymptotic distribution in PM. Extensive simulations show that our asymptotic distribution provides better approximations to the finite sample distribution. With a larger initial condition, one may be able to extend PM's asymptotic theory so that the modified asymptotic distribution depends on the initial condition. However, such a new asymptotic distribution depends on some nuisance parameters which cannot be consistently estimated. In the double asymptotic theory developed in the present study, the nuisance parameters are either known by the setting of the data structure or consistently estimable, making pivotal limit theory possible in continuous time models.

Our study also closely relates to the continuous time literature developed in statistics. In this literature, the least squares (LS) estimator, which is based on the Euler approximation to continuous time models, is often used to estimate the persistence parameter, when only discrete observations are available. The discretization error introduced by the Euler approximation has some important implications. For example, in developing the simultaneous double asymptotics for the stationary case, an extra condition that governs the relative convergence rates of N and h is needed in order to control the size of the discretization error; see, for example, Shimizu (2009) and Hu and Long (2009). In the explosive case, Shimizu (2009) showed that no asymptotic distribution can be derived because the size of the discretization error cannot be well controlled any more. However, our estimator of the persistence parameter is constructed directly from the exact discretized model, and hence, not subject to the discretization error. Consequently, in the stationary case, there is no need to impose an extra condition to control the joint behavior of N and h. More importantly, in the explosive case, we can derive a double asymptotic distribution for our estimator.

The rest of the paper is organized as follows. Section 2 introduces the model, builds the connection between our model and the mildly explosive process of PM, and discusses the relationship between our results and those in the literature. Section 3 develops the simultaneous double asymptotics. Two types of sequential double asymptotics are established in Section 4. In Section 5, we use simulated data to check the performance of our asymptotic theory and use real data to highlight the implications of our theory for statistical inference. Section 6 concludes. All the proofs are collected in the Appendix.

2. The model and the estimator

2.1. The model

The model studied in the paper is an Ornstein–Uhlenbeck (OU) diffusion process of the form:

$$dy(t) = \kappa\left(\mu - y(t)\right)dt + \sigma dL(t), \qquad y(0) = y_0 = O_p(1), \quad (2.1)$$

where $L(t)$ is a Lévy process defined on a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\right)$ with $L(0) = 0$ *a.s.*, $\mathcal{F}_t = \sigma\left\{[y(s)]_{s=0}^{t}\right\}$, and satisfies the following three properties:

1. Independent increments: for every increasing sequence of times $t_0, \ldots, t_n$ the random variables $L(t_0), L(t_1) - L(t_0), \ldots, L(t_n) - L(t_{n-1})$ are independent;
2. Stationary increments: the law of $L(t+h) - L(t)$ is independent of t;
3. Stochastic continuity: for any $\varepsilon > 0$, $t \geq 0$, $\lim_{h\to 0} P(|L(t+h) - L(t)| \geq \varepsilon) = 0$.[3]

The initial value y_0 is assumed to be independent of $L(t)$.

[2] Phillips (1987a) established the in-fill asymptotics for the unit root case (i.e., setting the persistence parameter κ to be zero) to take into account the effect of the initial condition in the limiting distribution. In Phillips (1987b) the in-fill asymptotics were established for the case where $\kappa \neq 0$ and the initial condition is set to be zero.

[3] This property allows existence of jumps happening at random times in sample path. As an effective way to introduce discontinuity into sample path, various Lévy processes have been developed in the asset pricing literature; see, for example, Barndorff-Nielsen (1998), Madan et al. (1998), and Carr and Wu (2003).