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Sample quantile analysis for long-memory stochastic volatility models

Hwai-Chung Ho *
Academia Sinica, Taiwan
National Taiwan University, Taiwan

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ABSTRACT

This study investigates asymptotic properties of sample quantile estimates in the context of long-memory stochastic volatility models in which the latent volatility component is an exponential transformation of a linear long-memory time series. We focus on the least absolute deviation quantile estimator and show that while the underlying process is a sequence of stationary martingale differences, the estimation errors are asymptotically normal with the convergence rate which is slower than $\sqrt{n}$ and determined by the dependence parameter of the volatility sequence. A non-parametric resampling method is employed to estimate the normalizing constants by which the confidence intervals are constructed. To demonstrate the methodology, we conduct a simulation study as well as an empirical analysis of the Value-at-Risk estimate of the S&P 500 daily returns. Both are consistent with the theoretical findings and provide clear evidence that the coverage probabilities of confidence intervals for the quantile estimate are severely biased if the strong dependence of the unobserved volatility sequence is ignored.

1. Introduction

Despite extensive studies of sample quantiles for stationary sequences, corresponding work pertaining to an important class of nonlinear time series remains largely absent. This paper aims to fill that gap by deriving the asymptotic distribution of quantile estimates for the long-memory stochastic volatility (LMSV) model. The development of this class of time series can be traced to two well-established stylized facts about speculative returns, namely, volatility clustering and Taylor's effect. The former concerns the phenomenon of a return series that large changes tend to be followed by large changes – of either signs – and small changes tend to be followed by small changes (Mandelbrot, 1963; Fama, 1965). The latter reports a positive and persistent autocorrelation in some nonlinear transformations of the returns, such as the square, the logarithm of square, and the absolute value, whereas the return series itself behaves like a white noise and contains very little correlation (e.g. Taylor, 1986; Ding et al., 1993). Stationary models proposed to describe these two properties include the ARCH (or GARCH) family (e.g. Engle, 1982; Bollerslev, 1986) and the stochastic volatility (SV) model (e.g. Taylor, 1986; Harvey et al., 1994). In all of these models the component that governs the volatility of the underlying return sequence is restricted to the case in which its autocorrelation function decays exponentially or, more generally, is summable, falling into the class of stationary time series usually described as short-memory. Lobato and Savin (1998) examine the absolute value and the square of the daily S&P 500 returns and show that the autocorrelations of the two transformed series decay hyperbolically at a rate of less than 1 and are thus not summable. Time series with such autocorrelation functions are called long-memory or long-range dependent. Following Lobato and Savin (1998), Breidt et al. (1998) extends the SV model to the LMSV model, which allows the latent volatility component to be long-memory, and show that the LMSV model better fits the decay rate of the autocorrelations of returns volatility than do some other commonly used models.

The LMSV model specified by Breidt et al. (1998) is as follows:

$$Y_t = \mu + e_t, \quad e_t = \sigma_t u_t \quad \text{and} \quad \sigma_t = \bar{\sigma} e^{Z_t/2}, \tag{1}$$

where $\bar{\sigma} > 0$, μ is the mean of Y_t, $\{u_t\}$ is i.i.d. with zero mean, unit variance, and independent of the latent volatility component $\{Z_t\}$. Furthermore, $\{Z_t\}$ is a linear process defined as

$$Z_t = \sum_{i=1}^{\infty} a_i \varepsilon_{t-i}, \qquad a_i \sim c^* \cdot i^{-\beta}, \tag{2}$$

* Correspondence to: Institute of Statistical Science, Academia Sinica, Taipei 115, Taiwan.
E-mail address: hcho@stat.sinica.edu.tw.

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where the i.i.d. innovations $\{\varepsilon_i\}$ have zero mean and finite variance σ_ε^2, c^* is some positive constant, $\beta \in (1/2, 1)$, and $g_n \sim h_n$ signifying $\lim_{n\to\infty} g_n/h_n = 1$. The term "long-memory" refers to the property of $\sum_i |a_i| = \infty$ or the fact that the autocovariance function $\gamma(j)$ of $\{Z_t\}$ is not summable because $\gamma(j) \sim c^{*2} j^{-(2\beta-1)}$. Note that we do not assume the Gaussianity of ε_i. One popular model of the long-memory linear process in Eq. (2) is the fractional autoregressive integrated moving average (ARFIMA) process of Adenstedt (1974), Granger and Joyeux (1980) and Hosking (1981). The traditional SV model, i.e. the short-memory version of Eq. (1), requires that the coefficient sequence $\{a_i\}$ be summable. Some general properties of this model can be found in Taylor (1986) and Harvey et al. (1994). In the sequel we denote by $F_e(\cdot)$, $F_u(\cdot)$, $F_Y(\cdot)$, and $F_Z(\cdot)$ the distribution functions of e_t, u_t, Y_t, and Z_t, respectively.

The long-memory process $\{Z_t\}$ is sometimes referred to as an $I(d)$ process with the memory parameter $d = 1-\beta$ (e.g. Brockwell and Davis, 1991). The LMSV model described in Eqs. (1) and (2) exhibits the desired property that $\{Y_t\}$ is white noise and $\{Y_t^2\}$ is long-memory. Because of this characteristic property, one needs to be careful in making statistical inference for the LMSV model if the statistics of interest involve nonlinear transformations of the underlying sequence. Take the estimation of the returns' Sharpe ratio for example, the square transformation is applied to the returns in order to estimate the standard deviation. As a result, while the returns form a martingale difference sequence, the persistent correlation of the volatility component emerges along with the estimation of Sharpe ratio, causing the standard root-n central limit theorem to fail (Ho, 2006). The similar non-standard asymptotics also occurs in the case of quantile estimates where the transformation involves the indicator function. There have been extensive studies on the asymptotic properties of sample quantiles for the sequence of short-memory random variables. The limiting distribution for the estimate $\hat{\xi}_0(\tau)$ of the τth quantile $\xi_0(\tau)$, which is usually derived with the help of the Bahadur representation (Bahadur, 1966), is

$$\sqrt{n}\left(\hat{\xi}_0(\tau) - \xi_0(\tau)\right) \xrightarrow{d} N\left(0, \sigma^2(\tau)\left[F_Y'(\xi_0(\tau))\right]^{-2}\right), \quad (3)$$

where the limiting variance $\sigma^2(\tau)$ is $\tau(1-\tau)$ for an iid sequence, and $\sigma^2(\tau) = \tau(1-\tau) + 2\sum_{k=1}^{\infty}\gamma_I(k)$ with $\gamma_I(k) = \mathrm{cov}(I(Y_1 < \xi_0(\tau)), I(Y_{1+k} < \xi_0(\tau)))$ if a certain type of weak dependence is imposed (Sen, 1972; Yoshihara, 1995). Wu (2005) establishes the Bahadur representation for linear processes which are allowed to be of short or long memory characterized by their summability conditions on the innovation coefficients. Because the similar representation is yet to be developed for the LMSV process, we employ a different method to study the issue of quantile estimation.

In the present paper the quantile estimator we choose is based on the absolute deviation loss and defined as the solution to the following minimization problem (see, Koenker and Bassett, 1978; Koenker, 2005):

$$\begin{aligned}\hat{\xi}_0(\tau) &= \underset{\theta\in\mathbb{R}}{\operatorname{argmin}}\left\{\tau\sum_{Y_t-\theta\geq 0}|Y_t-\theta| + (1-\tau)\sum_{Y_t-\theta<0}|Y_t-\theta|\right\}\\ &= \underset{\theta\in\mathbb{R}}{\operatorname{argmin}}\left\{\sum_{t=1}^{n}\rho_\tau(Y_t-\theta)\right\},\end{aligned} \quad (4)$$

where $\xi_0(\tau)$ is the τth quantile of F_Y, $\rho_\tau(x)$ (usually called the check function) is given by $x(\tau - I(x<0))$, $I(x<0)$ is the indicator function of $\{x<0\}$ and n denotes the sample size. As proposed in Theorem 2, the quantile estimates derived in Eq. (4) for the LMSV model obey the following non-root-n central limit theorem:

$$n^{1-H}\left(\hat{\xi}_0(\tau) - \xi_0(\tau)\right) \xrightarrow{d} N\left(0, \frac{(\xi_0(\tau)-\mu)^2\sigma^{*2}}{4}\right), \quad (5)$$

with $H = 3/2 - \beta = 1/2 + d$[1] and σ^{*2} being the limiting variance of $\sum_{i=1}^{n} Z_t/n^H$. The rate n^{1-H} given in Eq. (5) reveals that the statistical inferences made for $\xi_0(\tau)$ using the estimate $\hat{\xi}_0(\tau)$ would be greatly biased if the persistence in memory carried in the latent volatility component is not taken into account. For example, the width of the confidence interval constructed by using Eq. (5) for $\xi_0(\tau)$ becomes much greater than Eq. (3). This result has an important practical implication for the estimation of Value-at-Risk (VaR), a quantity commonly used in financial economics as a quantitative measure of investment risk. The VaR is defined as the maximal loss of an asset or portfolio with a given probability over a fixed period of time, or equivalently, VaR is a quantile of the loss distribution of the underlying asset or portfolio with a fixed time frame. Although Taylor's effect is widely recognized among researchers and practitioners, previous works (e.g. Dowd, 2001; Chen and Tang, 2005) on sample quantiles of returns seldom consider the models taking into account the fact that the autocorrelations of the returns' volatility decay very slowly. In light of the success of the LMSV time series in modeling financial returns, the result of Eq. (5) should offer a more reliable assessment for the variation of VaR estimates.

To use Eq. (5) to construct confidence intervals for quantile estimates, the usually unknown memory parameter H and the limiting variance σ^{*2} pose a challenging problem since they both depend on the unobservable sequence $\{Z_t\}$. We employ the sampling window method (see, for example, Hall et al., 1998; Zhang et al., 2013) to estimate the normalizing constants $n^{1-H}\cdot\sigma^{*2}$ by focusing on the log-square $\{\log Y_t^2\}$ of the observations.

The basic model (BM) defined in (1) is characterized by the exponential transformation and the independence between the volatility, $\{\sigma_t\}$, and the shock to the logarithms of prices, $\{u_t\}$. Among many other SV models considered in the literature, we briefly compare three of them with the BM; a more comprehensive and detailed accounts of the development of the SV model can be found in Taylor (1994) and Shephard (1996). The first model is the hidden Markov model (HMM) in which the volatility σ_t is a Markov chain with finite number of states. It is an intuitive model to describe the stylized fact of volatility clustering, but less successful in describing returns than models having continuous distribution for volatility. A review of the model and some theoretical results are provided in Hamilton (1994, Chapter 22). Second, the asymmetric SV model (ASVM) that allows some dependence between $\{\sigma_t\}$ and $\{u_t\}$ while maintaining that the returns $\{Y_t\}$ still form a martingale difference sequence. This can be achieved, for example, by supposing that u_t is independent of $\{\sigma_t, \sigma_{t-1}, \sigma_{t-2}, \ldots, u_{t-1}, u_{t-2}, \ldots\}$ (Taylor, 2005, Chapter 11). The ASVM is motivated by the property of asymmetric volatility (or the leverage effect) held by many speculative returns that the volatility of positive returns is in average less than that of negative returns. The similar effect of asymmetric volatility can also be generated by the EGARCH model introduced in Nelson (1991). Third, Robinson (2001) considers a different nonlinear model (referred as NM later) using more general functions instead of the exponential transformation. Specifically, the excess return in (1) is $e_t = f_1(\zeta_{1t})f_2(\zeta_{2t})$, where, for the univariate case, f_1 and f_2 are both square-integrable with respect to the normal distribution, and $\{(\zeta_{1t}, \zeta_{2t})\}$ is a sequence of bivariate normal random vectors. The normality assumption is the major constraint to the NM. The BM we focus on has some advantages over the aforementioned three types of SV models. It is simple and able to model important stylized facts about returns. Extending the BM to the long-memory case as shown in (2) is straightforward, but it is unclear as how

[1] Hereafter, both H and d appear interchangeably to denote the memory parameter.