ARTICLE IN PRESS
Journal of Econometrics ▮ (▮▮▮▮) ▮▮▮–▮▮▮
Contents lists available at ScienceDirect
Journal of Econometrics
journal homepage: www.elsevier.com/locate/jeconom

Asymptotic inference in multiple-threshold double autoregressive models

Dong Li [a], Shiqing Ling [b], Jean-Michel Zakoïan [c,*]

[a] Yau Mathematical Sciences Center and Center for Statistical Science, Tsinghua University, Beijing 100084, China
[b] Hong Kong University of Science and Technology, Hong Kong
[c] University Lille 3 and CREST, 15 boulevard Gabriel Péri, 92245 Malakoff Cedex, France

ARTICLE INFO

Article history:
Available online xxxx

JEL classification:
C13
C22

Keywords:
Compound Poisson process
Ergodicity
Quasi-maximum likelihood estimation
Strict stationarity
MTDAR model
Score test

ABSTRACT

This paper investigates a class of multiple-threshold models, called Multiple Threshold Double AR (MTDAR) models. A sufficient condition is obtained for the existence and uniqueness of a strictly stationary and ergodic solution to the first-order MTDAR model. We study the Quasi-Maximum Likelihood Estimator (QMLE) of the MTDAR model. The estimated thresholds are shown to be n-consistent, asymptotically independent, and to converge weakly to the smallest minimizer of a two-sided compound Poisson process. The remaining parameters are $\sqrt{n}$-consistent and asymptotically multivariate normal. In particular, these results apply to the multiple threshold ARCH model, with or without AR part, and to the multiple threshold AR models with ARCH errors. A score-based test is also presented to determine the number of thresholds in MTDAR models. The limiting distribution is shown to be distribution-free and is easy to implement in practice. Simulation studies are conducted to assess the performance of the QMLE and our score-based test in finite samples. The results are illustrated with an application to the quarterly US real GNP data over the period 1947–2013.

1. Introduction

Tong's (1978) threshold autoregressive (TAR) models have been extensively investigated in the literature and are arguably the most popular class of nonlinear time series models for the conditional mean. For financial time series, however, the conditional mean modeling has to be completed by a specification of the conditional variance. Indeed, typical effects such as the volatility clustering or the leverage effect have been widely documented in the empirical finance literature and such effects cannot be captured with independent innovations. For the conditional variance, the popularity of GARCH-type models, both in applied and theoretical works, has always increased since their introduction by Engle (1982). See, for example, Francq and Zakoïan (2010) for an overview on GARCH models. When the GARCH model is not directly applied to observations, but rather to the innovations of linear or nonlinear time series model, it can be more natural and convenient to specify the volatility as a function of the past observations rather than the past innovations. An example of such model is the double AR model introduced by Weiss (1984) and studied by Ling (2004, 2007).

In this article, we study the probabilistic properties and the estimation of a Multiple Threshold Double AR (MTDAR) model. More precisely, the model we consider in this article is the MTDAR(m; p) defined by

$$y_t = \sum_{i=1}^{m}\Big\{c_i + \sum_{j=1}^{p}\phi_{ij}y_{t-j} + \eta_t\Big(\omega_i + \sum_{j=1}^{p}\alpha_{ij}y_{t-j}^2\Big)^{1/2}\Big\} \times I\{y_{t-d} \in \mathcal{R}_i\}, \tag{1.1}$$

where m, p and d are positive integers, $c_i, \phi_{ij} \in \mathbb{R}$, $\omega_i > 0$, $\alpha_{ij} \geq 0$, the m sets $\mathcal{R}_i = (r_{i-1}, r_i]$ constitute a partition of the real line, $-\infty = r_0 < r_1 < \cdots < r_{m-1} < r_m = +\infty$, $I\{B\}$ denotes the indicator function of some event B, and $\{\eta_t\}$ is a sequence of independent and identically distributed (i.i.d.) random variables with zero mean. The standard ARCH(p) model can be obtained as a particular case by taking $c_i = \phi_{ij} = 0$ and $\alpha_{ij} = \alpha_j$ for all i and j, while a version of TAR(p) model is obtained by canceling the α_{ij}'s.

The first aim of this paper is to study the stability properties of the MTDAR(m; 1) model. The probabilistic structure of TAR models was studied by Chan et al. (1985), Chan and Tong (1985) and Tong (1990). Relying on the approach developed in the

* Corresponding author.
E-mail addresses: dongli@math.tsinghua.edu.cn (D. Li), maling@ust.hk (S. Ling), zakoian@ensae.fr (J.-M. Zakoïan).

http://dx.doi.org/10.1016/j.jeconom.2015.03.033
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book by Meyn and Tweedie (1996), we obtain explicit ergodicity conditions depending on the parameters of the extremal regimes (when $y_{t-d} \in \mathcal{R}_1$ or $y_{t-d} \in \mathcal{R}_m$) and the innovations distribution. A different approach was used by Cline and Pu (2004) who established sharp ergodicity conditions for a general class of threshold AR–ARCH models under assumptions we will discuss further.

The second aim of this article is to study the asymptotic properties of the Gaussian Quasi-Maximum Likelihood (QML) estimator of the vector of parameters, including the double-AR coefficients and the thresholds. The third aim of this article is to develop a score-based test to determine the number of thresholds in MTDAR models.

The literature on the estimation of threshold time-series models is vast. To cite but a few of such articles, let us mention Chan (1993), Hansen (2000), and more recently, Li and Ling (2012), Li et al. (2013a). These articles study the asymptotic properties of least-squares estimators (LSE) in threshold *linear* (AR or MA) models. Our framework is that of a threshold *nonlinear*[1] time-series model, for which a QML criterion allows to simultaneously estimate the conditional mean and variance. Simultaneous QML estimation of the conditional mean and variance was studied by Francq and Zakoian (2004) in the case of ARMA–GARCH, and by Meitz and Saikkonen (2011) for a general class of nonlinear AR–GARCH(1,1) models. A difference with these papers is that the conditional variance in Model (1.1) is specified in function of the observations rather than the innovations. Bardet and Wintenberger (2009) proved the asymptotic properties of the Gaussian QMLE for a general class of multidimensional causal processes, in which both the conditional mean and variance are specified as functions of the observations. However, their conditions for consistency and asymptotic normality require the existence of moments of orders 2 and 4, respectively, which we do not need for the class (1.1). Moreover, their assumptions rule out the possibility of thresholds in the parameter vector. To our knowledge, asymptotic results for estimation of nonlinear multiple threshold time series models had not yet been established in the literature.

The article is organized as follows. In Section 2, we study the existence of a strictly stationary and geometrically ergodic solution. In Section 3, we derive the asymptotic properties of the QML estimator. Some special cases of MTDAR models are analyzed in Section 4. Section 5 develops a score-based test to determine the number of thresholds in MTDAR models. Section 6 reports simulation results on the QMLE and the score test in finite samples. An empirical application is proposed in Section 7. All proofs of Theorems are displayed in the Appendix.

2. Stability properties of the MTDAR model

2.1. First-order model

We focus on the MTDAR(m; 1) model

$$y_t = \sum_{i=1}^{m} \Big(c_i + \phi_i y_{t-1} + \eta_t \sqrt{\omega_i + \alpha_i y_{t-1}^2} \Big) I\{y_{t-1} \in \mathcal{R}_i\}. \tag{2.1}$$

Without loss of generality, we assume in this section that $r_1 \le 0 \le r_{m-1}$. The aim of this section is to establish conditions for the existence of a strictly stationary and nonanticipative solution[2] to (2.1). We make the following assumption.

[1] nonlinearity having two causes: the thresholds and the presence of a volatility.

[2] A solution (y_t) is called nonanticipative if y_t can be written as a measurable function of $\{\eta_j : j \le t\}$.

A0: The distribution of η_t has a positive density f over $\mathbb{R}$. Moreover, $E|\eta_t|^s < \infty$ for some $s > 0$.

Let

$$\mu_1 = E \log|\phi_1 - \eta_t\sqrt{\alpha_1}|, \qquad p_1 = P(\eta_t < \phi_1/\sqrt{\alpha_1}),$$
$$\mu_m = E \log|\phi_m + \eta_t\sqrt{\alpha_m}|, \qquad p_m = P(\eta_t > -\phi_m/\sqrt{\alpha_m}).$$

By convention, $P(\eta_0 < a/b) = I\{a > 0\}$ if $b = 0$. Under **A0**, μ_1 and μ_m are well-defined but may be equal to $-\infty$ when $\phi_1 = 0$ or $\phi_m = 0$. We will prove the following result, using the approach developed by Meyn and Tweedie (1996) for establishing the geometric ergodicity of Markov chains.

Theorem 2.1. *Let Assumption* **A0** *hold and assume*

$$\gamma := \max\{(1+p_1)\mu_1 + (1-p_1)\mu_m, \\ (1-p_m)\mu_1 + (1+p_m)\mu_m\} < 0. \tag{2.2}$$

Then there exists a strictly stationary, nonanticipative solution $\{y_t\}$ to the MTDAR(m;1) Model (2.1) and the solution is unique and geometrically ergodic with $E|y_t|^u < \infty$ for some $u > 0$.

Remark 2.1. A simple sufficient condition for (2.2) is $\mu_1 < 0$ and $\mu_m < 0$. Note that the strict stationarity condition only depends on the coefficients of the two extremal regimes. This remarkable feature was obtained in the first-order multiple threshold AR model by Chan et al. (1985).

Remark 2.2. When the model is the multiple-threshold AR(1) model (or simply, when $\alpha_1 = \alpha_m = 0$), condition (2.2) reduces to

$$(0 < \phi_1 < 1,\ 0 < \phi_m < 1) \quad \text{or} \quad (\phi_1 < 0,\ \phi_m < 0,\ \phi_1\phi_m < 1)$$

or

$$(-1 < \phi_1\phi_m < 0,\ \phi_1 < 1,\ \phi_m < 1),$$

which is slightly stronger, when $\phi_1\phi_m < 0$, than the necessary and sufficient condition ($\phi_1 < 1, \phi_m < 1$ and $\phi_1\phi_m < 1$) established by Chan et al. (1985). For a standard AR(1) model ($\phi_1 = \phi_m$) we obtain the standard stationarity constraint $|\phi_1| < 1$. Fig. 1 gives the regions of (ϕ_1, ϕ_m) when $(\alpha_1, \alpha_m) = (0.1, 0.5), (1, 0.5), (1, 1)$ and $(1, 3)$ with $\eta_t \sim N(0, 1)$. We can see that ϕ_1, ϕ_m and $\phi_1\phi_m$ may be greater than 1, the upper boundary given by Chan et al. (1985) for TAR($1, m$) models.

Remark 2.3. When Model (2.1) reduces to a multiple threshold ARCH model, at least in its extremal regimes (i.e. $\phi_1 = \phi_m = 0$) with a symmetric density f, the condition (2.2) reduces to

$$\max\left(\alpha_1\alpha_m^3, \alpha_1^3\alpha_m\right) < \exp\{-4E\log\eta_t^2\}.$$

In particular, if $\alpha_1 = \alpha_m$, we retrieve the standard ARCH(1) condition: $\alpha_1 < \exp\{-E\log\eta_t^2\}$. When $\alpha_1 = \alpha_m = \alpha$ and $\phi_1 = \phi_m = \phi$ (which is in particular the case when $m = 1$), the condition (2.2) reduces to $\mu_1 + \mu_m < 0$, that is, $E\log|\phi^2 - \alpha\eta_t^2| < 0$. Moreover, if the distribution of η_t is symmetric, we then get the condition $\mu_1 = \mu_m < 0$, that is, $E\log|\phi - \eta_t\sqrt{\alpha}| < 0$. This is the necessary and sufficient strict stationarity condition obtained by Ling (2004) and Ling and Li (2008) for the double AR(1) model when η_t is normally distributed.

Remark 2.4. Cline and Pu (2004) obtained sharp ergodicity conditions for a general class of models encompassing Model (2.1), by an alternative approach called the *piggyback* method. From their Example 4.1, a condition for geometric ergodicity for our model is $(1-p_m)\mu_1 + (1-p_1)\mu_m < 0$ in our notations, which is in general a bit less restrictive than our condition (2.2). On the other hand, they need the assumption that $\sup_{x\in\mathbb{R}}\{(1+|x|)f(x)\} < \infty$ which we do not require.