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Semiparametric error-correction models for cointegration with trends: Pseudo-Gaussian and optimal rank-based tests of the cointegration rank

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ABSTRACT

This paper provides pseudo-Gaussian and locally optimal rank-based tests for the cointegration rank in linear cointegrated error-correction models with common trends and i.i.d. elliptical innovations. The proposed tests are asymptotically distribution-free, hence their validity does not depend on the actual distribution of the innovations. The proposed rank-based tests depend on the choice of scores, associated with a reference density that can freely be chosen. Under appropriate choices they are achieving the semiparametric efficiency bounds; when based on Gaussian scores, they moreover uniformly dominate their pseudo-Gaussian counterparts. Simulations show that the asymptotic analysis provides an accurate approximation to finite-sample behavior. The theoretical results are based on a complete picture of the asymptotic statistical structure of the model under consideration.

1. Introduction

Since their introduction by Granger (1981) and Engle and Granger (1981), cointegration models under error-correction form and the corresponding inference techniques have developed into a central topic in time-series econometrics, generating an extensive literature. The inferential side of that literature mainly deals with Gaussian, pseudo/quasi-Gaussian likelihood, or moment-based methods for problems related, e.g., to the cointegration rank or the cointegrating vectors; see, among many others, Stock (1987), Johansen and Juselius (1990), Johansen (1988, 1991, 1995), Phillips (1991), and Reinsel and Ahn (1992).

Whenever optimality issues – of a local and asymptotic nature in this context – are to be addressed, the adequate tool is Le Cam's asymptotic theory of statistical experiments; see, e.g., Strasser (1985), Le Cam (1986), Le Cam and Yang (1990), or van der Vaart (2000). The concept of *limit experiments* – more precisely, *limits of local sequences of experiments* – there plays an essential role: depending on their nature, those limit experiments indeed determine the asymptotic performances of tests and estimators, and the various efficiency bounds (parametric or nonparametric) that can be achieved. Often, they also suggest how to construct optimal procedures. That approach, for cointegration models, has been taken by several authors, including Phillips (1991), Jeganathan (1997)

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and Hodgson (1998a,b), and exploited to construct optimal tests for hypotheses on the cointegration vectors. It is worth already pointing out that we exploit the possible existence of trends in the data to improve the power of our test, and that in the absence of such trends our results reduce to those in Hodgson (1998b).

Based on a similar asymptotic approach, this paper focuses on the construction of optimal tests for hypotheses about the cointegration rank in error-correction models (ECMs) with possible deterministic linear trends—generated by non-zero values of the parameter μ in model (2.1) below. The presence of such (common) trends indeed has a dramatic impact on the nature of the various limit experiments. It leads, for specific directions in the parameter space, both within and outside the cointegrating space, to the familiar Locally Asymptotically Normal (LAN) structure, albeit with the nonstandard convergence rate $T^{3/2}$ (see Corollary 3.1). All possible limit experiments are characterized, in Proposition A.2 of the online supplementary appendix, for non-seasonal cointegrated ECMs with independent and identically elliptically distributed innovations. These limit experiments are generally of the complicated Locally Asymptotically Brownian Functional (LABF) type (Jeganathan, 1995). Considering, as a first step, the LAN subexperiment associated with perturbations of the cointegration rank only (no nuisances: all other parameters – cointegrating vectors and short-term dynamics – are supposed to be known), we construct new tests that are locally and asymptotically optimal (most stringent[1]) for the cointegration rank, under specified innovation density (Section 3.2). Invoking adaptivity arguments, we then show (Section 4) that those tests actually remain optimal when all other parameters are treated as nuisances to be estimated—that is, in the full experiment (still, under specified innovation density). The tests turn out to be of the Lagrange Multiplier type.

The incentive for including possible trends in the model actually originates in applications, and many empirical studies long ago have incorporated this possibility in their analyses. This is the case, for instance, of Bernard and Durlauf (1995) in their study of convergence and common trends in per capita output (see their Equation (3)). In the area of asset pricing, Nasseh and Strauss (2000) explicitly allow for the presence of deterministic time trends when studying the relation between stock prices and macroeconomic activity (see their Equation (3)). Swift (2011) documents a long-run relationship between health and GDP in OECD countries; model (1) in that paper explicitly allows for a parameter μ generating linear time trends. A more recent example is Wong et al. (2014), in a study of optimal investment with longevity risks (see their Equations (4)–(5)). When present, trends can and should be exploited, with huge potential benefits. It should be insisted, though, that, while the optimality properties of our tests very much depend on their presence, validity (in terms of asymptotic size) remains unaffected by their absence. As for the assumption of elliptically distributed innovations, it often has been considered in this context, see, for instance, Hodgson et al. (2002) and Hodgson and Vorkink (2003).

Another contribution of this paper is the introduction, in this multivariate time series context, of rank-based tests. The actual underlying density, in most applications, indeed remains unspecified, while the optimal parametric tests described in Section 3.2 typically lose their (asymptotic) validity under misspecified innovation densities. Pseudo-Gaussian (sometimes called Quasi Gaussian Maximum Likelihood, QMLE) methods then are the common practice. Therefore, we start (Proposition 4.1) with deriving pseudo-Gaussian versions of the optimal parametric tests of Section 3.2. Those pseudo-Gaussian tests are quite satisfactory[2] when actual densities are close to Gaussian ones. This, however, (due, e.g., to heavy tails) needs not be the case; and pseudo-Gaussian methods unfortunately may exhibit rather poor performances away from the Gaussian. Traditional semiparametric methods (in the Bickel et al. (1993) style) in principle provide the semiparametrically optimal solution in such cases. But they remain theoretically and numerically quite heavy, as they require guessing appropriate tangent space projections (unless the problem is *adaptive*), running kernel estimation of innovation densities, usually with sample splitting, etc. We propose avoiding this by turning to rank-based techniques.

General results by Hallin and Werker (2003) actually indicate that rank-based techniques offer an effective and numerically more tractable alternative to tangent space projections, achieving semiparametric efficiency (or parametric, in case the model is *adaptive*) at chosen,[3] reference densities. Accordingly, we introduce, in Section 4.3, a class of test statistics involving a multivariate notion of residual signed ranks. Those signed ranks – call them *elliptical ranks* – have been used, quite successfully, in a series of models in multivariate analysis and VARMA models with elliptical noise.[4] Following the methodology developed in those papers, we also construct optimal rank-based tests by projecting the optimal parametric test statistics of Section 3.2 onto those elliptical ranks, which in practice is quite easy.

We then show that the rank-based versions of the locally and asymptotically most stringent tests associated with the reference density still achieve parametric optimality under that reference density[5] while remaining (asymptotically) valid irrespective of the actual innovation density. Such rank-based tests offer several advantages. First of all, they are (asymptotically) distribution-free, so that their asymptotic critical values do not depend on the actual distribution of the innovations; were it not for the presence of estimated nuisance parameters, this distribution-freeness property would even hold exactly in finite samples. Second, while reaching parametric optimality under the reference density (which is not necessarily Gaussian), they often outperform, away from the reference density, sometimes quite significantly, the pseudo-Gaussian tests. In particular, the rank-based procedures associated with the Gaussian reference density (van der Waerden, or Gaussian-score tests) uniformly improve over the pseudo-Gaussian ones (see Section 4.4). In general, ranks[6] provide a form of robustness that stabilizes finite-sample sizes (see Section 5). The use of ranks relies on the assumed elliptical error distribution. Pseudo-Gaussian procedures do not require that assumption for validity. The present paper thus quantifies the efficiency gains that are possible in applications where ellipticity is likely to hold.

The remainder of this paper is organized as follows. Section 2 gives a precise description of the model and model assumptions

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[1] The concept of "most stringent" test boils down, in asymptotically Gaussian experiments, to the classical Lagrange multiplier tests based on quadratic forms. Section 11.9 in Le Cam (1986) provides a very accessible formal discussion of this concept, see in particular Lemma 1 and Corollary 2.

[2] They are not admissible, though, being uniformly dominated by the van der Waerden version of our rank-based tests: see Section 4.4.

[3] Possibly, via adequate data-driven methods.

[4] These include one-sample location (Hallin and Paindaveine, 2002a), serial independence (Hallin and Paindaveine, 2002b), linear models with VARMA errors (Hallin and Paindaveine, 2004a, 2005, 2006a), VAR order identification (Hallin and Paindaveine, 2004b), shape (Hallin and Paindaveine, 2006b; Hallin et al., 2006), homogeneity of scatter (Hallin and Paindaveine, 2008), principal and common principal components (Hallin et al., 2010, 2013, 2014).

[5] The problem, thus, is adaptive: the semiparametric and parametric efficiency bounds coincide.

[6] Note that ranks in the context of cointegration have been used before by Breitung (2001), in the spirit of Breitung and Gouriéroux (1997): no optimality concerns, and a totally different concept of ranks—instead of elliptical ranks computed from multivariate residuals, componentwise ranks, computed from the observations (p distinct rankings, thus, which are not distribution-free) are considered.