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A general method for third-order bias and variance corrections on a nonlinear estimator☆

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ABSTRACT

Motivated by a recent study of Bao and Ullah (2007a) on finite sample properties of MLE in the pure SAR (spatial autoregressive) model, a general method for third-order bias and variance corrections on a nonlinear estimator is proposed based on stochastic expansion and bootstrap. Working with concentrated estimating equation simplifies greatly the high-order expansions for bias and variance; a simple bootstrap procedure overcomes a major difficulty in analytically evaluating expectations of various quantities in the expansions. The method is then studied in detail using a more general SAR model, with its effectiveness in correcting bias and improving inference fully demonstrated by extensive Monte Carlo experiments. Compared with the analytical approach, the proposed approach is much simpler and has a much wider applicability. The validity of the bootstrap procedure is formally established. The proposed method is then extended to the case of more than one nonlinear estimator.

1. Introduction

Many econometric models share the following common features: (i) there is a nonlinear parameter that is the main source of bias in model estimation and main cause of difficulty in bias correction, (ii) there are many other parameters in the model but their estimates, given this nonlinear parameter, are either unbiased or can be easily bias-corrected, and (iii) the constrained estimates possess analytical expressions, leading to an analytical form for a concentrated estimating equation. These include the spatial autoregressive model, spatial panel model with fixed effects, dynamic regression model, dynamic panel model with fixed effects, Box–Cox regression, Weibull duration model, etc. The bias problem arising from the estimation of the nonlinear parameter has been widely recognized and a satisfactory treatment of it has been the main focus of many researchers in the last two decades (see, among others, Kiviet, 1995; Hahn and Kuersteiner, 2002; Hahn and Newey, 2004; Bun and Carree, 2005; Bao and Ullah, 2007a,b; Bao, 2013). Another important issue, the high-order correction on the variance of a bias-corrected estimator, has not been formally addressed.

Stochastic expansion (Rilstone et al., 1996; Ullah, 2004) is seen to be a very useful tool for studying the finite sample properties of a nonlinear estimator (Bao and Ullah, 2007a,b, 2009; Kundhi and Rilstone, 2008; Bao, 2013). However, in high-order bias and variance corrections: (i) it involves high dimension matrix manipulations and (ii) it requires closed form expressions of expectations of various quantities in the expansions, which are either very cumbersome to derive or simply do not even exist. We show in this paper that (i) can be overcome by focusing on the nonlinear parameter and working with the concentrated estimating equation, and (ii) can be overcome by a simple bootstrap procedure.

To illustrate the above ideas, consider first the spatial autoregressive (SAR) model:

$$Y_n = \lambda W_n Y_n + X_n \beta + \varepsilon_n, \quad \varepsilon_n = \sigma u_n, \tag{1}$$

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where Y_n is a vector of observations on n spatial units, X_n is an $n \times p$ matrix of values of p exogenous regressors, W_n is a specified $n \times n$ spatial weights matrix, ε_n is a vector of independent and identically distributed (iid) disturbances of zero mean and finite variance σ^2, λ is a scalar spatial parameter, and β is a $p \times 1$ vector of regression coefficients.[1]

Denote $\theta = \{\lambda, \beta', \sigma^2\}'$. The Gaussian log-likelihood function is,

$$\ell_n(\theta) = -\frac{n}{2}\log(2\pi\sigma^2) + \log|A_n(\lambda)| - \frac{1}{2\sigma^2}\left[A_n(\lambda)Y_n - X_n\beta\right]'\left[A_n(\lambda)Y_n - X_n\beta\right], \quad (2)$$

where $A_n(\lambda) = I_n - \lambda W_n$ and I_n is an $n \times n$ identity matrix. Maximizing $\ell(\theta)$ gives the maximum likelihood estimator (MLE) of θ if the errors are exactly normal, otherwise the quasi-MLE (QMLE). Given λ, the constrained QMLEs of β and σ^2 are

$$\hat{\beta}_n(\lambda) = (X_n'X_n)^{-1}X_n'A_n(\lambda)Y_n \quad \text{and}$$
$$\hat{\sigma}_n^2(\lambda) = n^{-1}Y_n'A_n'(\lambda)M_nA_n(\lambda)Y_n, \quad (3)$$

where $M_n = I_n - X_n(X_n'X_n)^{-1}X_n'$. These lead to the concentrated log-likelihood of λ as

$$\ell_n^c(\lambda) = -\frac{n}{2}[\log(2\pi) + 1] - \frac{n}{2}\log\hat{\sigma}_n^2(\lambda) + \log|A_n(\lambda)|. \quad (4)$$

Maximizing $\ell_n^c(\lambda)$ gives the unconstrained QMLE $\hat{\lambda}_n$ of λ. The unconstrained QMLEs of β and σ^2 are thus $\hat{\beta}_n \equiv \hat{\beta}_n(\hat{\lambda}_n)$ and $\hat{\sigma}_n^2 \equiv \hat{\sigma}_n^2(\hat{\lambda}_n)$. Write $\hat{\theta}_n = (\hat{\lambda}_n, \hat{\beta}_n', \hat{\sigma}_n^2)'$.

To study the finite sample properties of $\hat{\theta}_n$ following the stochastic expansion approach, one needs to derive analytically the expectations of various quantities involving derivatives of $\ell_n(\theta)$ (up to fourth order for third-order bias and variance corrections). While finding the expectations is not a problem in theory as it involves only quadratic forms of u_n, the dimensionality of the problem (up to $(p+2)^3 \times (p+2)$) greatly complicates the results that in turn hinders their practical tractability (see Bao, 2013, for a second-order bias formula). We note that if λ were known, then $\hat{\beta}_n(\lambda)$ is unbiased and $\hat{\sigma}_n^2(\lambda)$ can be made unbiased by multiplying a factor $n/(n-p)$. This suggests that in estimating the SAR model the main source of bias and the main difficulty in correcting the bias are associated with the estimation of λ. Lee (2007a) made a similar remark based on his Monte Carlo results. Further, given λ the finite sample variances of $\hat{\beta}_n(\lambda)$ and $\hat{\sigma}_n^2(\lambda)$ both possess explicit expressions. Thus, for bias and variance corrections for the SAR model it may be only necessary to focus on the estimation of λ. A multidimensional problem is thus reduced to a scalar one, which greatly simplifies the higher-order stochastic expansions. However, working with the concentrated log-likelihood $\ell_n^c(\lambda)$ makes the analytical derivation harder as it now involves ratios of quadratic forms (see Section 3 for details). Thus, for these expansions to be of a general practical value, they must be supplemented with simple ways for evaluating various expectations involving ratios of quadratic forms.

The above arguments extend directly to all other models of similar features as the SAR model. Take, for example, the Box–Cox transformation model (Box and Cox, 1964): $h(Y_n, \lambda) = X_n\beta + \sigma u_n$, where all quantities are defined similarly as the SAR model (1), except that h denotes a known nonlinear monotonic transformation indexed by an unknown transformation parameter λ, applied to Y_n elementwise. The concentrated log-likelihood of λ takes the form $\ell_n^c(\lambda) = -\frac{n}{2}[\log(2\pi) + 1] - \frac{n}{2}\log\hat{\sigma}_n^2(\lambda) + \sum_{i=1}^n \log h_y(Y_{n,i}, \lambda)$, where $\hat{\sigma}_n^2(\lambda) = n^{-1}h'(Y_n, \lambda)M_nh(Y_n, \lambda)$ and $h_y(Y_{n,i}, \lambda) = \partial h(Y_{n,i}, \lambda)/\partial Y_{n,i}$. It is clear that the analytical expectations of various quantities involving the derivatives of $\ell_n^c(\lambda)$ are not obtainable, and working with the full likelihood in this case does not solve this problem.

The above discussions show clearly the need for a general method for high-order bias and variance corrections that avoids the analytical derivations of various expectations, and thus works for all models even when the analytical expectations are not obtainable. Noting that the derivatives of $\ell_n^c(\lambda)$ for both the SAR model and the Box–Cox model discussed above can be expressed as functions of the parameter vector θ and the error vector u_n with iid elements, naturally, their expectations can be bootstrapped (see Efron, 1979).

In this paper we present a general method for third-order bias and variance corrections under a fairly general model specification that encompasses all the models mentioned above. The proposed approach is hybrid—combining stochastic expansion and bootstrap, with the former providing tractable approximations to the bias and variance (up to third-order) of a nonlinear estimator, and the latter making these expansions practically implementable. A key assumption followed in the literature is relaxed, resulting in different bias and variance formulas when concentrated estimating equation is used. The important issue: third-order correction on the standard error of a bias-corrected estimator, is formally studied.

When applied to the SAR model, the proposed approach quickly leads to a complete set of results for third-order bias and variance corrections, which extends Bao and Ullah (2007a) by (i) allowing regressors in the model, (ii) allowing nonnormal errors, and (iii) providing a third-order bias correction on $\hat{\lambda}_n$, and second- and third-order corrections on the variances of $\hat{\lambda}_n$ and the bias-corrected $\hat{\lambda}_n$. Compared with Bao (2013), where only a second-order bias formula for $\hat{\theta}_n$ is derived based on the full likelihood, our method can be viewed as a simpler alternative when only second-order bias correction on $\hat{\lambda}_n$ is concerned. In addition, our method provides a complete set of third-order results, including the third-order variance of the bias-corrected $\hat{\lambda}_n$. More importantly, the proposed approach is much simpler and has a much wider applicability than the analytical approach. The validity of the proposed bootstrap procedure is formally established, in general and under the SAR model. Finally, the method is extended to the models of more than one nonlinear parameter.

The rest of the paper is organized as follows. Section 2 presents the general method for third-order bias and variance corrections of a general nonlinear estimator. Section 3 presents the main theoretical results corresponding to the SAR model, followed by Monte Carlo results for the finite sample performance of the proposed method under the SAR model, where the effectiveness of the proposed method in correcting bias, variance, and hence in improving inference is fully demonstrated. Section 4 extends the proposed method to models of more than one nonlinear parameter. Section 5 concludes the paper.

2. A general method for bias and variance corrections

In this section, we first present revised third-order results by relaxing a key assumption, to suit the concentrated estimating equation, and then we introduce the bootstrap method for estimating quantities in the bias and variance formulas and prove its validity.

[1] For theory and applications, see Cliff and Ord (1973, 1981), Ord (1975), Anselin (1988, 2001), Case (1991), Case et al. (1993), Besley and Case (1995), Brueckner (1998), Anselin and Bera (1998), Kelejian and Prucha (1998, 1999, 2001), Bell and Bockstael (2000), Bertrand et al. (2000), Topa (2001), Lee (2002, 2003, 2004a, 2007a,b), Mynbaev and Ullah (2008), Robinson (2010), Su and Jin (2010), Su (2012), etc.