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Risk-parameter estimation in volatility models

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ABSTRACT

This paper introduces the concept of risk parameter in conditional volatility models of the form $\epsilon_t = \sigma_t(\theta_0)\eta_t$ and develops statistical procedures to estimate this parameter. For a given risk measure r, the risk parameter is expressed as a function of the volatility coefficients θ_0 and the risk, $r(\eta_t)$, of the innovation process. A two-step method is proposed to successively estimate these quantities. An alternative one-step approach, relying on a reparameterization of the model and the use of a non Gaussian QML, is proposed. Asymptotic results are established for smooth risk measures, as well as for the Value-at-Risk (VaR). Asymptotic comparisons of the two approaches for VaR estimation suggest a superiority of the one-step method when the innovations are heavy-tailed. For standard GARCH models, the comparison only depends on characteristics of the innovations distribution, not on the volatility parameters. Monte-Carlo experiments and an empirical study illustrate the superiority of the one-step approach for financial series.

1. Introduction

Modern financial risk management generally focuses on risks measures based on distributional information. Compared to traditional approaches relying on the marginal distribution of returns, more sophisticated approaches view risk as a stochastic process. For instance, conditional Value-at-Risk (VaR) – arguably the most widely used measure since the 1996 amendment of the Basel Capital Accord – is defined as the opposite of a quantile of the returns (or profit & losses, P&L, variables) conditional distribution. Another popular risk measure is the conditional Expected Shortfall which, conditional on the past returns, measures the average loss when the loss is above the VaR.[1] Many econometric approaches have been proposed in the finance and statistical literatures for measuring conditional risk.

A crucial issue that arises in this context is how to evaluate the performance of conditional risk estimators. Comparison of the performances of estimators of parameters based on the asymptotic theory is standard. But comparing the performances of VaR estimators, for instance, is more intricate because the conditional VaR is a random process, not a parameter.

The first objective of this paper is to introduce a concept of *risk parameter* in conditional volatility models. The risk parameter can be interpreted as a summary of conditional risk. Summaries of unconditional risk (such as the VaR based on historical simulation) are commonly used but they do not account for the dynamics of risk. By contrast, risk parameters are vector coefficients which take into account the returns dynamics and for which an asymptotic theory of estimation can be derived.

To be more specific, consider a conditional volatility model of the form

$$\epsilon_t = \sigma_t(\theta_0)\eta_t, \tag{1}$$

where ϵ_t denotes the log-return, σ_t is a volatility process, that is a positive measurable function of the past log-returns, θ_0 is a finite-dimensional parameter and (η_t) is a sequence of independent and identically distributed (iid) random variables, η_t being also independent of the past returns. Consider a risk measure, r, satisfying the assumption of positive homogeneity, such as the VaR or the Expected Shortfall. Then the conditional risk of ϵ_t is given by

$$r_{t-1}(\epsilon_t) = \sigma_t(\theta_0)r(\eta_t),$$

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[1] In the risk management literature, the term "conditional VaR" sometimes refer to what many authors, including us in this article, call Expected Shortfall. In this paper, we call conditional risks the risks computed conditional on the past returns.

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where $r(\eta_t)$ is a constant. In most parametric volatility models, multiplying the volatility by a constant amounts to modifying the parameter value. Under this assumption, we have

$$r_{t-1}(\epsilon_t) = \sigma_t(\theta_0^*), \quad \text{where } \theta_0^* = H\{\theta_0, r(\eta_t)\} \tag{2}$$

for some function H which is specific to the model under consideration. In this setting, we call θ_0^* the *risk parameter* associated to the risk function r. It incorporates not only the volatility parameters but also the (unconditional) risk of the innovation process (η_t). When r is the risk associated with the VaR at some level $\alpha \in (0, 1)$, the vector θ_0^* is referred to as the *VaR parameter* at level α.

Deriving an asymptotic theory for estimators of risk parameters is the second objective of this article. Two estimation procedures will be studied and compared. A two-step approach relies on the expression of θ_0^* in (2). Under the identifiability assumption

$$E\eta_t^2 = 1, \tag{3}$$

a consistent and asymptotically normal (CAN) estimator $\hat{\theta}$ of the parameter θ_0 can be obtained by standard methods for conditional volatility models, the most widely used being the Gaussian Quasi-Maximum Likelihood (QML). In a second step, an estimator $\hat{r}$ of the innovation risk $r(\eta_t)$ can be constructed, under conditions to be discussed, from the residuals $\hat{\eta}_t = \epsilon_t/\sigma_t(\hat{\theta})$ of the first step. A consistent estimator $H\{\hat{\theta}, \hat{r}\}$ of the risk parameter, θ_0^*, will be deduced (under smoothness assumptions on the function H). The asymptotic distribution of this estimator will follow from the joint asymptotic distribution of $\{\hat{\theta}, \hat{r}\}$.

An alternative strategy of estimation introduced in this article relies on a reparameterization of the conditional volatility model. The multiplicative form of model (1) generally allows us to rewrite it as

$$\epsilon_t = \sigma_t(\theta_0^*)\eta_t^*, \quad \text{with } r(\eta_t^*) = 1.$$

The latter equality replaces the standard assumption (3). The interest of such a representation is that, if a consistent estimator $\hat{\theta}_0^*$ of θ_0^* can be obtained, the conditional risk $r_{t-1}(\epsilon_t)$ of ϵ_t can be estimated in one step by $\sigma_t(\hat{\theta}_0^*)$.

Estimation of conditional volatility models under moment conditions different from (3) has been studied by Berkes and Horváth (2004), Zhu and Ling (2011), Francq and Zakoïan (2012) using non-Gaussian QML estimators. The poor performance of Gaussian QML in case of fat tails is nowadays a well-known issue and, to address it, researchers have developed some non-Gaussian extensions of QML for GARCH, where, through the trick of a scale parameter, consistent estimation is ensured. Fan et al. (2014), and Francq et al. (2011) proposed two-stage procedures based on non Gaussian QMLE for estimating standard GARCH models under the standard identifiability condition.

In the framework of this paper, the condition $r(\eta_t^*) = 1$ is not necessarily a moment condition. We propose a QML approach based on non-Gaussian densities depending on the risk function r. A case of particular importance is the VaR at a given level α: the identifiability condition consists in setting an appropriate quantile of the distribution of η_t^* to unity. It turns out that the only asymptotically valid QML criterion, that is, ensuring the consistency of the QML estimator of θ_0^* whatever the distribution of η_t^*, takes the form of a nonlinear quantile regression criterion.

The third objective of this article is to compare the one-step and two-step estimators of the VaR parameter. As we will see, the assumptions required for the CAN of the two estimators are quite different. When such assumptions are met, the asymptotic variances can be compared. Surprisingly, for important subclasses of conditional volatility models the ranking of the two methods, in terms of asymptotic efficiency, depends on α and on simple characteristics of the law of η_t, but not on the volatility parameter θ_0.

Most of the previous work on the statistical inference for GARCH-type models dealt with the estimation of volatility parameters. The asymptotic theory of the QML estimation for volatility parameters has been extensively studied, in particular for the GARCH(p, q) by Berkes et al. (2003) and Francq and Zakoïan (2004), for general models by Mikosch and Straumann (2006), Straumann and Mikosch (2006), Bardet and Wintenberger (2009). For the VaR parameter, it turns out that the QML criterion can be written under the form of a M-estimation criterion which is similar to those introduced in the quantile regression literature (see Koenker (2005) for a comprehensive book on quantile regression, and see Xiao and Koenker (2009), Xiao and Wan (2010) for recent applications to linear GARCH models) and in the least-absolute deviations (LAD) time series literature (see Davis et al. (1992), Davis and Dunsmuir (1997), Breidt et al. (2001), Ling (2005)).

Recent references dealing with estimation uncertainty in the evaluation of conditional risks measures are Spierdijk (forthcoming) and Yan et al. (2013). The latter paper proposes a new quantile estimator based on adaptive estimation, while Spierdijk proposes a method based on residual subsample bootstrap.

The paper is organized as follows. In Section 2 we introduce the concept of risk parameter in a general conditional volatility model, and we discuss identifiability issues. Section 3 is devoted to the asymptotic properties of non-Gaussian QML estimators for general smooth risk measures r. Section 4 is devoted to the estimation of the VaR parameter. The smoothness assumptions introduced in Section 3 being non satisfied by the VaR, the asymptotic properties of the one-step estimator are established in a completely different manner. The asymptotic properties of the two-step method are also established, and are compared with those of the one-step estimator. A Monte-Carlo study and applications on real financial data are proposed in Section 5. Section 6 concludes. Proofs are collected in the Appendix.

2. Risk parameter in volatility models

Most conditional volatility models are of the form

$$\begin{cases} \epsilon_t = \sigma_t \eta_t \\ \sigma_t = \sigma(\epsilon_{t-1}, \epsilon_{t-2}, \ldots; \theta_0) \end{cases} \tag{4}$$

where (η_t) is a sequence of iid random variables, η_t being independent of $\{\epsilon_u, u < t\}$, $\theta_0 \in \mathbb{R}^m$ is a parameter belonging to a parameter space Θ, and $\sigma : \mathbb{R}^\infty \times \Theta \to (0, \infty)$. When $E\eta_t = 0$ and $E\eta_t^2 = 1$, the variable σ_t^2 is generally referred to as the volatility of ϵ_t. However, we will not make such moment assumptions in this section and the following ones. A leading model, the most widely used among practitioners, is the GARCH(1, 1) model defined by

$$\sigma_t^2 = \omega_0 + \alpha_0\epsilon_{t-1}^2 + \beta_0\sigma_{t-1}^2, \tag{5}$$

where $\theta_0 = (\omega_0, \alpha_0, \beta_0)' \in (0, \infty)\times[0, \infty)\times[0, 1)$. For this model we have $\sigma_t^2 = \sum_{i=1}^{\infty} \beta_0^{i-1}(\omega_0 + \alpha_0\epsilon_{t-i}^2)$, which is of the form (4).

It is assumed throughout that

A0: There exists a function H such that for any $\theta \in \Theta$, for any $K > 0$, and any sequence $(x_i)_i$

$$K\sigma(x_1, x_2, \ldots; \theta) = \sigma(x_1, x_2, \ldots; \theta^*), \quad \text{where } \theta^* = H(\theta, K).$$

Assumption **A0** means that the volatility model is stable by scaling. All commonly used conditional volatility models satisfy this assumption, as can be seen from Table 1. See Sucarrat and Escribano (2010), Wintenberger (2013) for recent references on the log-GARCH and EGARCH, and Francq and Zakoïan (2010) for references