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Pricing with finite dimensional dependence

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ABSTRACT

We consider derivative pricing in factor models, where the factor is Markov with Finite Dimensional Dependence (FDD). The FDD condition allows for explicit formulas for derivative prices and their term structure and in this respect is a serious competitor of models with affine dynamic factors. The approach is illustrated by a comparison of the prices of realized and integrated volatility swaps. We show that the usual practice of replacing a payoff written on the realized volatility by the payoff written on the integrated volatility can imply pricing errors which are not negligible when the volatility of the volatility is large.

1. Introduction

The predictive properties of nonlinear state space models with finite dimensional dependence (FDD) have been analyzed in Gourieroux and Jasiak (2000). They consider a (multidimensional) Markov process (y_t) and the (nonlinear) predictor space defined as the following set of functions of y_{t-1}:

$$Pred = \{E[g(y_t)|y_{t-1}] = E_{t-1}g(y_t), g \text{ varying}\}. \tag{1.1}$$

They prove that the predictor space has a finite dimension if and only if the transition[1] of the process can be decomposed as:

$$p(y_t|y_{t-1}) = l_0(y_t)\sum_{k=1}^{K} a_k(y_t)b_k(y_{t-1}) = l_0(y_t)a'(y_t)b(y_{t-1}), \text{ say}, \tag{1.2}$$

where l_0 is a benchmark density function (with respect to an appropriate dominating measure μ), and $[b_1(y_{t-1}), \ldots, b_K(y_{t-1})]$ is a basis of the predictor space. The decomposition into l_0, a, b is not unique, but the predictor space is always identifiable.[2] According to the type of application the benchmark density can be the stationary density of the process, if it exists, or simply be equal to one, when the dominating measure is the Lebesgue measure.

The aim of our paper is the derivative pricing in (multivariate) factor models, where the factor is Markov with finite dimensional dependence.

The pricing formulas are derived in Section 2. We first consider European derivatives written on factor y. We get closed form expressions for their prices and their term structure. We also explain how the factor process can be a FDD Markov process in both the historical and risk-neutral worlds, and discuss the relationship between the associated historical and risk-neutral decompositions. The pricing formulas are extended to European derivatives written on cumulated factor transforms. The volatility and variance swaps have typically such a payoff and their prices are discussed in detail. The models with smooth transition are special FDD Markov processes with hidden regimes. They are introduced in Section 3. We provide regime interpretation of the associated term structure of predictions in both historical and risk-neutral worlds. The results are illustrated by the example of gamma mixtures in Section 4 to show the flexibility of FDD models for describing the term structure patterns and their dynamics. We see that the FDD term

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[1] Decomposition (1.2) looks like a nonlinear canonical representation of the transition pdf (Lancaster, 1968). However,
(i) the dimension of the predictor space is finite, whereas it is infinite in general in the canonical representation;
(ii) l_0 is not necessarily the marginal density of y_t;
(iii) We have not introduced the orthogonality conditions of the type $\int a_k(y)a_j(y) l_0(y)d\mu(y) = \int b_k(y)f_j(y)l_0(y)d\mu(y) = 0, \forall k \neq j$.
Thus the a_k (resp. b_k) functions cannot be interpreted as orthogonal canonical directions.

[2] When $K = 2$ and the factor is one-dimensional, the joint distribution of (y_t, y_{t-1}) belongs to the Sarmanov family of bivariate distributions (Sarmanov, 1966), and (Huang and Lin, 2011).

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structure models can easily feature several bumps, which is especially convenient for the analysis of ultra-long riskfree bonds and of corporate bonds. We compare in Section 5 the pricing of realized and integrated variance swaps. Indeed the variance swaps proposed on the market have a payoff written on a realized variance, but are often priced as if the payoff were written on the (unobservable) integrated stochastic volatility. We discuss the magnitude of the error in such an approximated pricing formula and show that it cannot be neglected if the volatility of the volatility is large. Section 6 concludes. Proofs are gathered in Appendices.

2. The pricing formulas

Let us consider a discrete time model with a FDD Markov (multidimensional) state process (y_t) and a historical transition satisfying decomposition (1.2). The information of the investor at date t includes the current and lagged values of the state process. We denote by $m(y_{t+1})$ the stochastic discount factor (sdf) for period $(t, t+1)$ and $E_0(G) = \int G(y)l_0(y)d\mu(y)$ the expectation of a matrix function $G(y)$ with respect to the benchmark distribution $l_0(y)d\mu(y)$, where μ is a dominating measure.

2.1. European derivatives

The price at date t of a European derivative paying $g(y_{t+H})$ at $t+H$ is:

$$\Pi(t, H; g) = E_t[m(y_{t+1}) \dots m(y_{t+H})g(y_{t+H})]. \tag{2.1}$$

This price admits a closed form expression (see Appendix).

Proposition 1. $\Pi(t, H; g) = E_0(mga')[E_0(mba')]^{H-1}b(y_t), H \geq 1$.

The derivative prices are linear combinations of the generators $b_k(y_t), k = 1, \dots, K$ of the predictor space at date t. Moreover, at any given date t and for any payoff function g, the term structures of derivative prices:

$H \to \Pi(t, H; g)$ are generated by the baseline term structures, elements of: $H \to [E_0(mba')]^H$.

For instance, by taking $g(y) = 1$, we get the term structure of zero-coupon bond prices:

$$B(t, H) = E_0(ma')[E_0(mba')]^{H-1}b(y_t), \quad H \geq 1, \tag{2.2}$$

which is linear in the factor $F_t = b(y_t)$, which is a nonlinear function of state (y_t). This linearity property of the zero-coupon prices has to be distinguished from the linearity property of the yields appearing in the affine term structure models (see e.g. Duffie and Kan, 1996) the zero-coupon prices.

Under the FDD dynamics, the term structure of the zero-coupon prices is a combination of the baseline term structures defined by the elements of matrix $[E_0(mba')]^{H-1}$. These baseline term structures depend on exponential functions λ_j^H corresponding to the eigenvalues of $E_0(mba')$, and possibly on exponentials multiplied by polynomials if the algebraic multiplicity of λ_j is strictly smaller than its geometric multiplicity, that is the dimension of its associated eigenspace. Thus, model (2.2) provides a justification of the specification considered in Vasicek and Fong (1982), or Shea (1985), for exponential splines with random coefficients, or in McCulloch (1975), for exponentials times polynomials specification.

2.2. Historical versus risk-neutral factor dynamics

The historical prediction of payoff $g(y_{t+1})$ is obtained in the special case $m(y) = 1$. We get:

$$E_t[g(y_{t+H})] = E_0(ga')[E_0(ba')]^{H-1}b(y_t). \tag{2.3}$$

Therefore the term structure of risk premia can be defined as:

$$\Pi(t, H; g) - E_t[g(y_{t+H})]B(t, H) = \{E_0(mga')[E_0(mba')]^{H-1} - B(t, H)E_0(ga')[E_0(ba')]^{H-1}\}b(y_t) \tag{2.4}$$

where $B(t, H)$ is given by formula (2.2).

The risk-neutral transition of the factor is given in the next proposition.

Proposition 2. *The risk-neutral transition of factor y is:*

$$q(y_t|y_{t-1}) = l_0(y_t)\sum_{k=1}^{K} a_k^*(y_t)b_k^*(y_{t-1}),$$

where:

$$a_k^*(y_t) = [a_k(y_t)m(y_t)]/E_0(a_k m),$$

$$b_k^*(y_{t-1}) = [b_k(y_{t-1})E_0(a_k m)] \Big/ \left\{\sum_{j=1}^{K} E_0(a_j m)b_j(y_{t-1})\right\}.$$

Proof. We have:

$$\begin{aligned} q(y_t|y_{t-1}) &= \frac{m(y_t)p(y_t|y_{t-1})}{\int m(y_t)p(y_t|y_{t-1})d\mu(y_t)} \\ &= \frac{l_0(y_t)\sum_{k=1}^{K}[a_k(y_t)m(y_t)b_k(y_{t-1})]}{\sum_{j=1}^{K}\{E_0(a_j m)b_j(y_{t-1})\}} \\ &= l_0(y_t)\sum_{k=1}^{K}[a_k^*(y_t)b_k^*(y_{t-1})]. \quad \square \end{aligned}$$

Thus, the factor process is a FDD Markov process under both the historical and risk-neutral distributions, with a same dimension K of the historical and risk-neutral predictor spaces. This result is a consequence of the particular choice of the sdf, which depends on the current factor value of the factor only. Also note that functions a^*, b^* have been normalized in the decomposition. This normalization will be useful for the hidden regime interpretation in Section 3.

2.3. Derivatives written on cumulated factor transforms

Let us now consider the price of a derivative paying:

$c[g(y_{t+H^*}) + \dots + g(y_{t+H})], \quad 1 \leq H^* \leq H,$

at date $t+H$. Its price is:

$$\Pi(t, H^*, H; c, g) = E_t\{m(y_{t+1}) \dots m(y_{t+H})c[g(y_{t+H^*}) + \dots + g(y_{t+H})]\}. \tag{2.5}$$

As usual these prices are easy to derive if function c is exponential $c(z) = \exp(uz)$, say, where argument u may be complex. By considering complex argument u, we allow for the use of transform analysis to deduce the prices of derivatives with other real c functions (Duffie et al., 2000). For exponential function c, we get:

$$\Pi(t, H^*, H; \exp(u.), g) = E_t\{m(y_{t+1}) \cdots m(y_{t+H}) \exp[ug(y_{t+H^*}) + \dots + ug(y_{t+H})]\}. \tag{2.6}$$

Proposition 3.

$$\Pi(t, H^*, H; \exp(u.), g) = E_0[m\exp(ug)a'][E_0(m\exp(ug)ba')]^{H-H^*}[E_0(mba')]^{H^*-1}b(y_t),$$

for $1 \leq H^* \leq H$.