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A test for second order stationarity of a multivariate time series

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ABSTRACT

It is well known that the discrete Fourier transforms (DFTs) of a second order stationary time series between two distinct Fourier frequencies are asymptotically uncorrelated. In contrast for a large class of second order nonstationary time series, including locally stationary time series, this property does not hold. In this paper these starkly differing properties are used to define a global test for stationarity based on the DFT of a vector time series. It is shown that the test statistic under the null of stationarity asymptotically has a chi-squared distribution, whereas under the alternative of local stationarity asymptotically it has a noncentral chi-squared distribution. Further, if the time series is Gaussian and stationary, the test statistic is pivotal. However, in many econometric applications, the assumption of Gaussianity can be too strong, but under weaker conditions the test statistic involves an unknown variance that is extremely difficult to directly estimate from the data. To overcome this issue, a scheme to estimate the unknown variance, based on the stationary bootstrap, is proposed. The properties of the stationary bootstrap under both stationarity and nonstationarity are derived. These results are used to show consistency of the bootstrap estimator under stationarity and to derive the power of the test under nonstationarity. The method is illustrated with some simulations. The test is also used to test for stationarity of FTSE 100 and DAX 30 stock indexes from January 2011–December 2012.

1. Introduction

In several disciplines, as diverse as finance and the biological sciences, there has been a dramatic increase in the availability of multivariate time series data. In order to model this type of data, several multivariate time series models have been proposed, including the Vector Autoregressive model and the vector GARCH model, to name but a few (see, for example, Lütkepohl, 2005 and Laurent et al., 2012). The majority of these models are constructed under the assumption that the underlying time series is stationary. For some time series this assumption can be too strong, especially over relatively long periods of time. However, relaxing this assumption, to allow for nonstationary time series models, can lead to complex models with a large number of parameters, which may not be straightforward to estimate. Therefore, before fitting a time series model, it is important to check whether or not the multivariate time series is second order stationary.

Over the years, various tests for second order stationarity for univariate time series have been proposed. These include, Priestley and Subba Rao (1969), Loretan and Phillips (1994), von Sachs and Neumann (1999), Paparoditis (2009, 2010), Dahlhaus and Polonik (2009), Dwivedi and Subba Rao (2011), Dette et al. (2011), Dahlhaus (2012, Example 10), Jentsch (2012), Lei et al. (2012) and Nason (2013). However, as far as we are aware there does not exist any tests for second order stationarity of multivariate time series (Jentsch, 2012 does propose a test for multivariate stationarity, but the test was designed to detect the alternative of a multivariate periodically stationary time series). One crude solution is to individually test for stationarity for each of the univariate processes. However, there are a few drawbacks with this approach. The first is that most multiple testing schemes use a Bonferroni correction, which results in a test statistic which is extremely conservative. The second problem is that such a strategy can lead to misleading conclusions. For example if each of the marginal time series are second order stationary, but the cross-covariances are second order nonstationary, the above testing scheme would not be able to detect the alternative. Therefore there is a need to develop a test for stationarity of a multivariate time series, which is the aim in this paper.

The majority of the univariate tests, are local, in the sense that they are based on comparing the local spectral densities over

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various segments. This approach suffers from some possible disadvantages. In particular, the spectral density may locally vary over time, but this does not imply that the process is second order nonstationary, for example Hidden Markov models can be stationary but the spectral density can vary according to the regime. For these reasons, we propose a global test for multivariate second order stationarity.

Our test is motivated by the tests for detecting periodic stationarity (see, for example, Goodman, 1965, Hurd and Gerr, 1991, Bloomfield et al., 1994 and Olhede and Ombao, 2013) and the test for second order stationarity proposed in Dwivedi and Subba Rao (2011), all these tests use fundamental properties of the discrete Fourier transform (DFT). More precisely, the above mentioned periodic stationarity tests are based on the property that the discrete Fourier transform is correlated if the difference in the frequencies is a multiple of $2\pi/P$ (where P denotes the periodicity), whereas Dwivedi and Subba Rao (2011) use the idea that the DFT asymptotically uncorrelates stationary time series, but not nonstationary time series. Motivated by Dwivedi and Subba Rao (2011), in this paper, we exploit the uncorrelating property of the DFT to construct the test. However, the test proposed here differs from Dwivedi and Subba Rao (2011) in several important ways, these include (i) our test takes into account the multivariate nature of the time series, (ii) the test proposed here is defined such that it can detect a wider range of alternatives and (iii) the test in Dwivedi and Subba Rao (2011) assumes Gaussianity or linearity of the underlying time series (and calculates the power under the assumption of Gaussianity), which in several econometric applications is unrealistic, whereas our test allows for testing of nonlinear stationary time series.

In Section 2, we motivate the test statistic by comparing the covariance between the DFT of stationary and nonstationary time series, where we focus on the large class of nonstationary processes called locally stationary time series (see Dahlhaus, 1997, Dahlhaus and Polonik, 2006 and Dahlhaus, 2012 for a review). Based on these observations, we define DFT covariances which in turn are used to define a Portmanteau-type test statistic. Under the assumption of Gaussianity, the test statistic is pivotal, however for non-Gaussian time series the test statistic involves a variance which is unknown and extremely difficult to estimate. If we were to ignore this variance (and thus implicitly assume Gaussianity) then the test can be unreliable. Therefore in Section 2.4 we propose a bootstrap procedure, based on the stationary bootstrap (first proposed in Politis and Romano, 1994), to estimate the variance. In Section 3, we derive the asymptotic sampling properties of the DFT covariance. We show that under the null hypothesis, the mean of the DFT covariance is asymptotically zero. In contrast, under the alternative of local stationarity, we show that the DFT covariance estimates nonstationary characteristics in the time series. These results are used to derive the sampling distribution of the test statistic. Since the stationary bootstrap is used to estimate the unknown variance, in Section 4, we analyze the stationary bootstrap when the underlying time series is stationary and nonstationary. Some of these results may be of independent interest. In Section 5 we show that under (fourth order) stationarity the bootstrap variance estimator is a consistent estimator of the true variance. In addition, we analyze the bootstrap variance estimator under nonstationarity and show that it has an influence on the power of the test. The test statistic involves some tuning parameters and in Section 6.1, we give some suggestions on how to select these tuning parameters. In Section 6.2, we analyze the performance of the test statistic under both the null and the alternative and compare the test statistic when the variance is estimated using the bootstrap and when Gaussianity is assumed. In the simulations we include both stationary GARCH and Markov switching models and for nonstationary models we consider time-varying linear models and the random walk. In Section 6.3, we apply our method to analyze the FTSE 100 and DAX 30 stock indexes. Typically, stationary GARCH-type models are used to model this type of data. However, even over the relatively short period January 2011–December 2012, the results from our test suggest that the log returns are nonstationary.

The proofs can be found in the Appendix.

2. The test statistic

2.1. Motivation

Let us suppose $\{\underline{X}_t = (X_{t,1}, \ldots, X_{t,d})', t \in \mathbb{Z}\}$ is a d-dimensional constant mean, multivariate time series and we observe $\{\underline{X}_t\}_{t=1}^T$. We define the vector discrete Fourier transform (DFT) as

$$\underline{J}_T(\omega_k) = \frac{1}{\sqrt{2\pi T}} \sum_{t=1}^{T} \underline{X}_t e^{-it\omega_k}, \quad k = 1, \ldots, T,$$

where $\omega_k = 2\pi \frac{k}{T}$ are the Fourier frequencies. Suppose that $\{\underline{X}_t\}$ is a second order stationary multivariate time series, where the autocovariance matrices of $\{\underline{X}_t\}$ satisfy

$$\sum_{h=-\infty}^{\infty} |h| \cdot |\mathrm{cov}(X_{h,j_1}, X_{0,j_2})| < \infty \quad \text{for all } j_1, j_2 = 1, \ldots, d. \tag{2.1}$$

It is well known for $k_1 - k_2 \neq 0$, that $\mathrm{cov}(J_{T,m}(\omega_{k_1}), J_{T,n}(\omega_{k_2})) = O(\frac{1}{T})$ (uniformly in T, k_1 and k_2), in other words the DFT has transformed a stationary time series into a sequence which is approximately uncorrelated. The behavior in the case that the vector time series is second order *nonstationary* is very different. To obtain an asymptotic expression for the covariance between the DFTs, we will use the rescaling device introduced by Dahlhaus (1997) to study locally stationary time series, which is a class of nonstationary processes. $\{\underline{X}_{t,T}\}$ is called a locally second order stationary time series, if its covariance structure changes slowly over time such that there exist smooth matrix functions $\{\boldsymbol{\kappa}(\cdot; r)\}_r$ which can approximate the time-varying covariance matrices. More precisely, $|\mathrm{cov}(\underline{X}_{t,T}, \underline{X}_{\tau,T}) - \boldsymbol{\kappa}(\frac{\tau}{T}; t-\tau)|_1 \leq T^{-1}\kappa(t-\tau)$, where $\sum_h \kappa(h) < \infty$. An example of a locally stationary model which satisfies these conditions is the time-varying moving average model defined in Dahlhaus (2012, Eqs. (63)–(65)) (with $\ell(j) = \log(|j|)^{1+\varepsilon}|j|^2$ for $|j| \neq 0$). It is worth mentioning that Dahlhaus (2012) uses the slightly weaker condition $\ell(j) = \log(|j|)^{1+\varepsilon}|j|$. In the Appendix (Lemma A.8), we show that

$$\mathrm{cov}(\underline{J}_T(\omega_{k_1}), \underline{J}_T(\omega_{k_2})) = \int_0^1 \mathbf{f}(u; \omega_{k_1}) \exp(-i2\pi u(k_1 - k_2)) du + O\left(\frac{1}{T}\right), \tag{2.2}$$

uniformly in T, k_1 and k_2, where $\mathbf{f}(u; \omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \boldsymbol{\kappa}(u; h) \times \exp(-ih\omega)$ is the local spectral density matrix (see Lemma A.8 for details). We recall if $\{\underline{X}_t\}_t$ is second order stationary then the 'spectral density' function $\mathbf{f}(u; \omega)$ does not depend on u and the above expression reduces to $\mathrm{cov}(\underline{J}_T(\omega_{k_1}), \underline{J}_T(\omega_{k_2})) = O(\frac{1}{T})$ for $k_1 - k_2 \neq 0$. It is interesting to observe that for locally stationary time series its DFT sequence mimics the behavior of a time series, in the sense that the correlation between the DFTs decays the further apart the frequencies.

A further, related motivation for our test is that a time series $\{\underline{X}_t\}$ is second order stationary if and only if it admits the Fourier–Stieltjes integral (Cramér Representation)

$$\underline{X}_t = \int_0^{2\pi} \exp(it\omega) d\underline{Z}(\omega), \tag{2.3}$$

where $\{\underline{Z}(\omega); \omega \in [0, 2\pi]\}$ is an orthogonal increment vector process (see for example, Yaglom (1987), Chapter 2). The DFT $\underline{J}_T(\omega_k)$