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Nonlinear regressions with nonstationary time series

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ABSTRACT

This paper develops asymptotic theory for a nonlinear parametric cointegrating regression model. We establish a general framework for weak consistency that is easy to apply for various nonstationary time series, including partial sums of linear processes and Harris recurrent Markov chains. We provide limit distributions for nonlinear least square estimators, extending the previous works. We also introduce endogeneity to the model by allowing the error to be serially dependent on and cross correlated with the regressors.

1. Introduction

The past few decades have witnessed significant developments in cointegration analysis. In particular, extensive researches have focused on cointegration models with linear structure. Although it gives data researchers convenience in implementation, the linear structure is often too restrictive. In particular, nonlinear responses with some unknown parameters often arise in the context of economics. For empirical examples, we refer to Granger and Teräsvirta (1993) as well as Teräsvirta et al. (2011). In this situation, it is expected that the nonlinear cointegration model captures the features of many long-run relationships in a more realistic manner.

A typical nonlinear parametric cointegrating regression model has the form

$$y_t = f(x_t, \theta_0) + u_t, \quad t = 1, \ldots, n, \tag{1}$$

where $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ is a known nonlinear function, x_t and u_t are regressor and regression errors, and θ_0 is an m-dimensional true parameter vector that lies in the parameter set Θ. With the observed nonstationary data $\{y_t, x_t\}_{t=1}^n$, this paper is concerned with the nonlinear least square (NLS) estimation of the unknown parameters $\theta_0 \in \Theta$. In this regard, Park and Phillips (2001) (PP henceforth) considered x_t to be an integrated, $I(1)$, process. Based on PP framework, Chang et al. (2001) introduced an additional linear time trend term and stationary regressors into the model (1). Chang and Park (2003) extended it to nonlinear index models driven by integrated processes. More recently, Choi and Saikkonen (2010), Gao et al. (2009) and Wang and Phillips (2012) developed statistical tests for the existence of a nonlinear cointegrating relation. Park and Chang (2011) allowed the regressors x_t to be contemporaneously correlated with the regression errors u_t and Shi and Phillips (2012) extended the model (1) by incorporating a loading coefficient.

The present paper has a similar goal to the previously mentioned papers but offers some more general results, which have some advantages for empirical studies. First of all, we establish a general framework for weak consistency of the NLS estimator $\hat{\theta}_n$, allowing for the x_t to be a wider class of nonstationary time series. The set of sufficient conditions is easy to apply for various nonstationary regressors, including partial sums of linear processes and recurrent Markov chains. Furthermore, we provide limit distributions for the NLS estimator $\hat{\theta}_n$. It deserves to mention that the routine employed in this paper to establish the limit distributions of $\hat{\theta}_n$ is different from those used in the previous works, e.g. Park and Phillips (2001). Roughly speaking, our routine is related to the joint

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distributional convergence of a martingale under target and its conditional variance, rather than using classical martingale limit theorem which requires establishing the convergence in probability for the conditional variance. In nonlinear cointegrating regressions, there are some advantages for our methodology since it is usually difficult to establish the convergence in probability for the conditional variance, in particular, in the situation that the regressor x_t is a nonstationary time series. Second, in addition to the commonly used martingale innovation structure, our model allows for serial dependence in the equilibrium errors u_t and the innovations driving x_t. It is important as our model permits joint determination of x_t and y_t, and hence the system is a time series structural model. Under such situation, the weak consistency and limit distribution of the NLS estimator $\hat{\theta}_n$ are also established.

This paper is organized as follows. Section 2 presents our main results on weak consistency of the NLS estimator $\hat{\theta}_n$. Theorem 2.1 provides a general framework. Its applications to integrable and non-integrable f are given in Theorems 2.2–2.5, respectively. Section 3 investigates the limit distributions of $\hat{\theta}_n$ in which the model (1) has a martingale structure. Extension to endogeneity is presented in Section 4. As mentioned above, our routine establishing the limit distribution of $\hat{\theta}_n$ is different from previous works. Section 5 performs simulation, and discusses the numerical values of means and standard errors which provide the evidence of accuracies of our NLS estimator. Section 6 presents an empirical example, providing a link between our theory and real applications. The model of interest is the carbon Kuznets curve relating the per capita CO_2 emissions and per capita GDP. Endogeneity occurs in this example due to potential misreporting of GDP, omitted variable bias and reverse causality. Section 7 concludes the paper. Section 8 provides partial technical proofs. Full details of the technical proofs can be found in the supplemental material of this paper (see Appendix A), where we also provide a unit root test for our empirical example, and other details for simulation.

Throughout the paper, we denote constants by $C, C_1, C_2, \ldots$ which may be different at each appearance. For a vector $x = (x_1, \ldots, x_m)$, assume that $\|x\| = (x_1^2 + \cdots + x_m^2)^{1/2}$, and for a matrix A, the norm operator $\|\cdot\|$ is defined by $\|A\| = \sup_{x:\|x\|=1} \|xA\|$. Furthermore, the parameter set $\Theta \subset \mathbb{R}^m$ is assumed to be compact and convex, and the true parameter vector θ_0 is an interior point of Θ.

2. Weak consistency

This section considers the estimation of the unknown parameters θ_0 in model (1) by NLS. Let $Q_n(\theta) = \sum_{t=1}^n (y_t - f(x_t, \theta))^2$. The NLS estimator $\hat{\theta}_n$ of θ_0 is defined to be the minimizer of $Q_n(\theta)$ over $\theta \in \Theta$, that is,

$$\hat{\theta}_n = \arg\min_{\theta\in\Theta} Q_n(\theta), \tag{2}$$

and the error estimator is defined by $\hat{\sigma}_n^2 = n^{-1} \sum_{t=1}^n \hat{u}_t^2$, where $\hat{u}_t = y_t - f(x_t, \hat{\theta}_n)$. To investigate the weak consistency for the NLS estimator $\hat{\theta}_n$, this section assumes the regression model (1) to have a martingale structure. In this situation, our sufficient conditions are closely related to those of Wu (1981), Lai (1994) and Skouras (2000), intending to provide a general framework. In comparison to the papers mentioned, our assumptions are easy to apply, particularly in nonlinear cointegrating regression situation as stated in the two examples below. Extension to endogeneity between x_t and u_t is investigated in Section 4.

2.1. A framework

We make use of the following assumptions for the development of weak consistency.

Assumption 2.1. For each $\pi, \pi_0 \in \Theta$, there exists a real function $T : \mathbb{R} \to \mathbb{R}$ such that

$$|f(x, \pi) - f(x, \pi_0)| \le h(\|\pi - \pi_0\|)\, T(x), \tag{3}$$

where $h(x)$ is a bounded real function such that $h(x) \downarrow h(0) = 0$, as $x \downarrow 0$.

Assumption 2.2. (i) $\{u_t, \mathcal{F}_t, 1 \le t \le n\}$ is a martingale difference sequence satisfying $E(|u_t|^2|\mathcal{F}_{t-1}) = \sigma^2$ and $\sup_{1\le t\le n} E(|u_t|^{2q}|\mathcal{F}_{t-1}) < \infty$ a.s., where $q > 1$; and
(ii) x_t is adapted to $\mathcal{F}_{t-1}$, $t = 1, \ldots, n$.

Assumption 2.3. There exists an increasing sequence $0 < \kappa_n \to \infty$ such that

$$\kappa_n^{-2} \sum_{t=1}^n [T(x_t) + T^2(x_t)] = O_P(1), \tag{4}$$

and for any $0 < \eta < 1$ and $\theta \ne \theta_0$, where $\theta, \theta_0 \in \Theta$, there exist $n_0 > 0$ and $M_1 > 0$ such that

$$P\left(\sum_{t=1}^n (f(x_t, \theta) - f(x_t, \theta_0))^2 \ge \kappa_n^2/M_1\right) \ge 1 - \eta, \tag{5}$$

for all $n > n_0$.

Theorem 2.1. *Under Assumptions 2.1–2.3, the NLS estimator $\hat{\theta}_n$ is a consistent estimator of θ_0, i.e. $\hat{\theta}_n \to_P \theta_0$. If in addition $\kappa_n^2 n^{-1} = O(1)$, then $\hat{\sigma}_n^2 \to_P \sigma^2$, as $n \to \infty$.*

Assumptions 2.1 and 2.2 are the same as those used in Skouras (2000), which are standard in the NLS estimation theory. Also see Wu (1981) and Lai (1994). Assumption 2.3 is used to replace (3.8), (3.9) and (3.11) in Skouras (2000), in which some uniform conditions are used. In comparison to Skouras (2000), our Assumption 2.3 is related to the conditions on the regressor x_t and is more natural and easy to apply. In particular, it is directly applicable in the situation that T is integrable and the regressor x_t is a nonstationary time series, as stated in the following sub-section.

2.2. Assumption 2.3: integrable functions

Due to Assumption 2.1, $f(x, \theta) - f(x, \theta_0)$ is integrable in x if T is an integrable function. This class of functions includes $f(x, \theta_1, \theta_2) = \theta_1 |x|^{\theta_2} I(x \in [a, b])$, where a and b are finite constants, the Gaussian function $f(x, \theta) = e^{-\theta x^2}$, the Laplacian function $f(x, \theta) = e^{-\theta|x|}$, the logistic regression function $f(x, \theta) = e^{\theta|x|}/(1 + e^{\theta|x|})$, etc. In this sub-section, two commonly used nonstationary regressors x_t are shown to satisfy Assumption 2.3 if T is integrable.

Example 1 (*Partial Sums of Linear Processes*). Let $x_t = \sum_{j=1}^t \xi_j$, where $\{\xi_j, j \ge 1\}$ is a linear process defined by

$$\xi_j = \sum_{k=0}^\infty \phi_k\, \epsilon_{j-k}, \tag{6}$$

where $\{\epsilon_j, -\infty < j < \infty\}$ is a sequence of i.i.d. random variables with $E\epsilon_0 = 0$, $E\epsilon_0^2 = 1$ and the characteristic function $\varphi(t)$ of ϵ_0 satisfying $\int_{-\infty}^\infty |\varphi(t)|dt < \infty$. The coefficients ϕ_k are assumed to satisfy one of the following conditions:

C1. $\phi_k \sim k^{-\mu}\rho(k)$, where $1/2 < \mu < 1$ and $\rho(k)$ is a function slowly varying at ∞.

C2. $\sum_{k=0}^\infty |\phi_k| < \infty$ and $\phi \equiv \sum_{k=0}^\infty \phi_k \ne 0$.