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Nonparametric tests for tail monotonicity

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This article proposes nonparametric tests for tail monotonicity of bivariate random vectors. The test statistic is based on a Kolmogorov–Smirnov-type functional of the empirical copula. Depending on the serial dependence features of the data, we propose two multiplier bootstrap techniques to approximate the critical values. We show that the test is able to detect local alternatives converging to the null hypothesis at rate $n^{-1/2}$ with a non-trivial power. A simulation study is performed to investigate the finite-sample performance and finally the procedure is illustrated by testing intergenerational income mobility and testing a market data set.

1. Introduction

Economists are often confronted with relationships between variables where, a priori, a monotone dependence stands at reason. One may think for instance of expenditures versus income at household levels, wages versus cognitive skills or sons' income versus parental income, see, e.g., Lee et al. (2009) for these and other well-known economic relationships. As an example from finance, the liquidity preference hypothesis states that the expected return on government securities is a monotonically increasing function of remaining time to maturity.

In many instances, one is interested in testing for the a priori hypothesis that a given relationship is monotone. Not surprisingly, in recent years, some effort has been made to develop different testing procedures for this hypothesis, see, e.g., Lee et al. (2009), Patton and Timmermann (2010), Delgado and Escanciano (2012) and Gijbels and Sznajder (2013). The proposed tests differ either methodologically or, more substantially, in the exact formulation of the hypothesis of monotonicity. A common concept is the one of *(increasing) stochastic monotonicity*, also known as *positive regression dependence*: a random variable Y is positively regression dependent on X, notationally $\mathrm{PRD}(Y|X)$, if

$$x \mapsto F_{Y|X=x}(y) = \Pr(Y \le y \mid X = x) \text{ is non-increasing for all } y \in \mathbb{R}.$$

Roughly, $\mathrm{PRD}(Y|X)$ means that, for any $x_1 < x_2$, it is more likely to observe large realizations of Y for $X = x_2$ than for $X = x_1$. Both Delgado and Escanciano (2012) and Lee et al. (2009) developed tests for this hypothesis and applied them to test for monotonicity between sons' incomes and parental incomes. In Section 4, we will reconsider this data set.

Of course, PRD is not the only possibility of specifying a monotone relationship between random variables. Indeed, it might be quite restrictive in certain applications. For instance, let X and Y denote the failure times of two components in a reliability system, e.g., of two machines in a production process which support each other. A positive (monotone) dependence between the failure times seems to be suggestive; however, $\mathrm{PRD}(Y|X)$ might fail in such a system: if the machines work permanently over time, but in the case of failure can only be repaired during the weekdays, then a failure of the X-machine on friday could have a better effect on the survival time of the Y-machine than a failure of the X-machine on the successional saturday morning.

Another example for a non-PRD, yet "monotone" relationship is given by net wages against gross wages. Net wages will usually fail to be positively regression dependent on gross wages if the rate of taxation increases at specific levels.

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Early work by Fama (1984) on the afore-mentioned liquidity preference hypothesis showed some evidence that expected returns on term premiums for T-bills are not monotonic. Even though the results were disputed by other authors (McCulloch, 1987; Richardson et al., 1992; Boudoukh et al., 1999), the discussion might have been profited from a test for a weaker concept of monotonicity between random variables.

Such weaker concepts of monotonicity can for instance be defined by slight changes of the objects in the definition of PRD($Y|X$). The concepts of (positive) tail monotonicity, dating back to Lehmann (1966) and Esary and Proschan (1972), read as follows:

- Y is left tail decreasing in X if the function $x \mapsto \Pr(Y \le y \mid X \le x)$ is non-increasing for all $y \in \mathbb{R}$. Notation: LTD($Y|X$);
- Y is right tail increasing in X if the function $x \mapsto \Pr(Y > y \mid X > x)$ is non-decreasing for all $y \in \mathbb{R}$. Notation: RTI($Y|X$).

As shown in, e.g., Nelsen (2006) or Joe (1997), stochastic monotonicity implies tail monotonicity. Belzunce et al. (2007) rephrased the concepts in terms of conditional distributions. In particular, a relationship between absolutely continuous random variables is RTI($Y|X$) if and only if $P^{[Y|X=x]}$ is stochastically dominated by $P^{[Y|X>x]}$ for all $x \in \mathbb{R}$, and it is PRD($Y|X$) if and only if $P^{[Y|X=x_1]}$ is stochastically dominated by $P^{[Y|X=x_2]}$ for all $x_1 \le x_2$. This rephrasing shows that both for the failure time example and for the net-gross wages example discussed above, right tail increasingness is a reasonable hypothesis.

The considerations in the preceding paragraphs form the main motivation of our work: provided a practitioner has found some evidence that a strong form of monotonicity (such as PRD) does not hold, he might be interested in testing procedures for the weaker concept of tail monotonicity. Additionally, there are some theoretical reasons which demand for such a test: Genest and Segers (2010) showed that, under tail monotonicity, estimation of the copula function by means of the empirical copula is more efficient if contingent additional knowledge of the marginal distribution function is completely ignored. Hence, a positive testing result on the question of tail decreasingness may result in a more efficient subsequent data-analysis. Moreover, our research is motivated by a recent paper by Kojadinovic and Yan (2012). These authors propose two nonparametric tests of exchangeability of a random vector which only work for left tail decreasing random vectors. Hence, prescience of left tail decreasingness is essential for their approach.

To the best of our knowledge, the only reference proposing a test for tail monotonicity seems to be a very recent publication by Gijbels and Sznajder (2013), in which, however, no explicit theory is provided to justify the critical values. Their procedure, which is completely different from our approach, is based on the constrained copula estimation and is restricted to the i.i.d. case. It is the purpose of our paper to fill these gaps and to develop a new testing procedure that works both for i.i.d. as well as for time series data, and to give precise proofs for results about the level and the consistency of the test.

The construction of our test statistic is based on a characterization of tail monotonicity by the copula function. For details, we refer the reader to Section 2 (see also Nelsen, 2006). In order to cope with serially dependent, strongly mixing data sets, we exploit a block multiplier bootstrap method, see Bücher and Ruppert (2013). By doing this, we also outline a direction on how to adapt other tests to the serially dependent setting, as, e.g., in Delgado and Escanciano (2012) on regression dependence and in Scaillet (2005) on the related hypothesis of *positive quadrant dependence*.

The remaining part of this article is organized as follows. In Section 2 we define the test for tail monotonicity and propose two multiplier bootstrap procedures for deriving critical values. We begin with the easier setting of serial independent data sets and proceed with a (block) multiplier bootstrap to deal with data sets descending from a strictly stationary, strongly mixing time series. In Section 3 we investigate the finite-sample performance of the test by means of a simulation study. The approach is illustrated by testing intergenerational income mobility and a market data set in Section 4. Finally, all proofs are deferred to Section 5.

2. Testing tail monotonicity

For the sake of a clear exposition we only consider the concept of left tail decreasingness in the subsequent part of this paper. We propose a nonparametric test for the hypothesis

H_0 : Y is left tail decreasing in X.

The construction of variants for the right tail (RTI) or for respective concepts of (negative) monotonicity (right tail decreasingness, RTD, or left tail increasingness, LTI, see, e.g., Nelsen, 2006, Section 5.2.2) is straightforward. The underlying idea of our test is a rephrasement of H_0 in terms of the copula of X and Y, which, in the case of continuity of X and Y, is the unique function C on $[0, 1]^2$ such that $\Pr(X \le x, Y \le y) = C\{\Pr(X \le x), \Pr(Y \le y)\}$ for all $x, y \in \mathbb{R}$, see Sklar (1959). A simple calculation (see, e.g., Nelsen, 2006) shows that Y is left tail decreasing in X if and only if

$$H_0 : u \mapsto C(u, v)/u = \Pr(G(Y) \le v \mid F(X) \le u) \text{ is a non-increasing function for each } v \in [0, 1], \tag{2.1}$$

where F and G denote the cumulative distribution functions (cdfs) of the continuous random variables X and Y, respectively. Set $\Delta = \{(s, t) \in [0, 1]^2 \mid s \le t\}$ and denote by $\mathcal{T} : \ell^\infty([0, 1]^2) \to \ell^\infty([0, 1] \times \Delta)$ the operator which maps a function $H : [0, 1]^2 \to \mathbb{R}$ to the function $\mathcal{T}(H) : [0, 1] \times \Delta \to \mathbb{R}$ defined by

$$\mathcal{T}(H)(u, s, t) = sH(t, u) - tH(s, u).$$

Here, $\ell^\infty(T)$ is defined as the set of all real-valued bounded functions on a set T. Now, non-increasingness of the function in (2.1) is obviously equivalent to the fact that

$$H_0 : \mathcal{T}(C)(u, s, t) \le 0 \quad \text{for all } u \in [0, 1], (s, t) \in \Delta.$$

This is the property we test for in this paper. Our approach is based on the fact that the copula C, and therefore also $\mathcal{T}(C)$, can be estimated by its sample analog C_n, the empirical copula. We check if a suitable functional of $\mathcal{T}(C_n)$ is sufficiently small in a certain way. Basing the test on the empirical copula results in invariance of the test with respect to strictly increasing transformations of the marginals which should be a minimal requirement for any test of tail monotonicity.

More precisely, suppose that $(X_1, Y_1), \dots, (X_n, Y_n)$ is a sample of a strictly stationary stochastic process $(X_i, Y_i)_{i \in \mathbb{Z}}$ with copula C and continuous marginal cumulative distribution functions F and G, respectively. Moreover, suppose that the process $(X_i, Y_i)_{i \in \mathbb{Z}}$ is strongly mixing with α-mixing coefficient $\alpha(r) = O(r^{-a})$ for $r \to \infty$ and some $a > 1$, where

$$\alpha(r) = \sup_{s \ge 0} \sup_{A \in \mathcal{F}_s, B \in \mathcal{F}^{s+r}} |\Pr(A \cap B) - \Pr(A)\Pr(B)|$$

and $\mathcal{F}_s = \sigma\{(X_i, Y_i) : i \le s\}$ and $\mathcal{F}^t = \sigma\{(X_i, Y_i) : i \ge t\}$. Note that for an i.i.d. sequence $(X_i, Y_i)_{i \in \mathbb{Z}}$ we have $\alpha(r) = 0$ for all $r \ge 1$.

To estimate the copula C we first transform the random variables X_i and Y_i to pseudo-observations of the copula by defining

$$\hat{U}_i = \frac{\text{rank of } X_i \text{ among } X_1, \dots X_n}{n+1},$$

$$\hat{V}_i = \frac{\text{rank of } Y_i \text{ among } Y_1, \dots Y_n}{n+1}.$$

Obviously, we have $\hat{U}_i = \frac{n}{n+1} F_n(X_i)$ and $\hat{V}_i = \frac{n}{n+1} G_n(Y_i)$, where $F_n(x) = n^{-1} \sum_{i=1}^n \mathbf{1}(X_i \le x)$ and $G_n(y) = n^{-1} \sum_{i=1}^n \mathbf{1}(Y_i \le y)$