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On the properties of the coefficient of determination in regression models with infinite variance variables[☆]

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ABSTRACT

We examine the asymptotic properties of the coefficient of determination, R^2, in models with α-stable random variables. If the regressor and error term share the same index of stability $\alpha < 2$, we show that the R^2 statistic does not converge to a constant but has a nondegenerate distribution on the entire [0, 1] interval. We provide closed-form expressions for the cumulative distribution function and probability density function of this limit random variable, and we show that the density function is unbounded at 0 and 1. If the indices of stability of the regressor and error term are unequal, we show that the coefficient of determination converges in probability to either 0 or 1, depending on which variable has the smaller index of stability, irrespective of the value of the slope coefficient. In an empirical application, we revisit the Fama and MacBeth (1973) two-stage regression and demonstrate that in the infinite-variance case the R^2 statistic of the second-stage regression converges to 0 in probability even if the slope coefficient is nonzero. We deduce that a small value of the R^2 statistic should not, in itself, be used to reject the usefulness of a regression model.

1. Introduction

Granger and Orr (1972) lead off their article " 'Infinite variance' and research strategy in time series analysis" by questioning the uncritical use of the normal distribution assumption in economic modeling and estimation:

> It is standard procedure in economic modeling and estimation to assume that random variables are normally distributed. In empirical work, confidence intervals and significance tests are widely used, and these usually hinge on the presumption of a normal population. Lately, there has been a growing awareness that some economic data display distributional characteristics that are flatly inconsistent with the hypothesis of normality.

Due importantly to the seminal work of Mandelbrot (1963), non-Gaussian α-stable distributions are often considered to provide the basis for more realistic distributional assumptions for some economic data, especially for high-frequency financial time series such as those of exchange rate fluctuations and stock returns. Financial time series are typically fat-tailed and excessively peaked around their mean—phenomena that can be better captured by α-stable distributions with $1 < \alpha < 2$ rather than by the normal distribution for which $\alpha = 2$. The α-stable distributional assumption with $\alpha < 2$ is a generalization of rather than a strict alternative to the Gaussian distributional assumption. If an economic series fluctuates according to an α-stable distribution with $\alpha < 2$, it is known that many of the standard methods of statistical analysis do not apply in the conventional way. In particular, as we demonstrate in this paper, if $\alpha < 2$ the coefficient of determination of a regression model has several nonstandard properties. Moreover, these properties are sufficiently important to cast doubt on the suitability of the coefficient of determination as a general goodness of fit criterion in regressions in which the regressor(s) and the error term

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are characterized by strong outlier activity, regardless of whether the sample value of the coefficient of determination is high or low.

The linear regression model is one of the most commonly used and basic econometric tools, not only for the analysis of macroeconomic relationships but also for the study of financial market data. Typical examples for the latter case are the estimation of the *ex-post* version of the capital asset pricing model (CAPM) and the two-stage modeling approach of Fama and MacBeth (1973). Because of the prevalence of heavy-tailed distributions in financial time series, it is of interest to study how regression models perform when the data are heavy-tailed rather normally distributed. There are many heavy-tailed distributions that could be considered. One such class of distributions that is particularly suitable in a regression model context is the class of α-stable distributions, because (i) these distributions are able to capture the relative frequencies of extreme vs. ordinary observations in economic and financial variables, (ii) they have the convenient statistical property of closure under convolution, and (iii) only α-stable distributions can serve as limiting distributions of normalized sums of independent and identically distributed (iid) random variables, as proven in Zolotarev (1986). The second and third properties are especially appealing for regression analysis because the disturbance term may often be interpreted as a random variable which represents the sum of all external effects not captured by the regressors.

In this paper, we show that infinite variance of the regressor and disturbance term has important consequences for the asymptotic properties of the coefficient of determination, R^2, a very frequently used goodness-of-fit measure. We show that if the regressor and error term are both α-stable (with $\alpha < 2$) with the same index of stability, the R^2 statistic does not converge to a fixed (positive) constant but has a nondegenerate limiting distribution on the (0, 1) interval. Hence, a low value of R^2 in an empirical regression application should not, by itself, be interpreted as implying either that the model is poorly specified or that there is no statistically significant (linear) relationship between the regressor and the dependent variable. In an empirical application, we revisit the Fama and MacBeth (1973) two-stage regression approach and establish that infinite variance of the regression variables affects decisively the interpretation of the well-known stylized empirical fact that the R^2 statistics in static CAPM models tend to be very close to 0. Specifically, we find that a low value of the R^2 statistic should *not* be used to conclude that the relationship between the regressor and the regressand is "flat".

The rest of our paper is structured as follows. In Section 2 we provide a brief summary of the properties of α-stable distributions and of aspects of estimation, hypothesis testing, and model diagnostic checking in regression models with α-stable variables. Section 3 provides a detailed analysis of the asymptotic properties of the coefficient of determination in regression models with infinite variance variables. Our empirical application is presented in Section 4, and Section 5 offers concluding remarks.

2. Framework

2.1. A brief overview of the properties α-stable distributions

A random variable X is said to have a stable distribution if, for any positive integer $n > 2$, there exist coefficients $a_n > 0$ and $b_n \in \mathbb{R}$ such that $X_1 + \cdots + X_n \overset{d}{=} a_n X + b_n$, where $X_1, \ldots, X_n$ are independent copies of X and $\overset{d}{=}$ signifies equality in distribution. The coefficients a_n are necessarily of the form $a_n = n^{1/\alpha}$ for some $\alpha \in (0, 2]$; see Feller (1971, Chapter VI.1). The parameter α is called the index of stability of the distribution, and a random variable X with index of stability α is called α-stable . An α-stable distribution is described by four parameters and will be denoted by $S(\alpha, \gamma, \beta, \delta)$. Closed-form expressions for the probability density functions (pdfs) of α-stable distributions are known to exist only for three special cases.[2] However, closed-form expressions for the characteristic functions of α-stable distributions are readily available. One parameterization of the logarithm of the characteristic function of $S(\alpha, \gamma, \beta, \delta)$ is[3]

$$\ln \mathrm{E}\, e^{i\tau X} = i\delta\tau - \gamma^{\alpha} |\tau|^{\alpha} \left(1 + i\beta \,\mathrm{sign}\,(\tau)\, \omega(\tau, \alpha)\right), \tag{1}$$

where sign (τ) equals -1 for $\tau < 0$, 0 for $\tau = 0$, and $+1$ for $\tau > 0$; and $\omega(\tau, \alpha)$ equals $-\tan(\pi\alpha/2)$ for $\alpha \neq 1$ and $(2/\pi) \ln |\tau|$ for $\alpha = 1$.[4]

The asymptotic tail shape of an α-stable distribution is determined by its index of stability $\alpha \in (0, 2]$. Skewness is governed by $\beta \in [-1, 1]$; the distribution is symmetric about $\delta \in \mathbb{R}$ if and only if $\beta = 0$. The scale and location parameters of α-stable distributions are denoted by $\gamma > 0$ and δ, respectively. If $\alpha = 2$, the right-hand side of Eq. (1) reduces to $i\delta\tau - \gamma^2\tau^2$, which is that of a Gaussian random variable with mean δ and variance $2\gamma^2$.

For $\alpha < 2$ and $|\beta| < 1$, the tails of an α-stable random variable X satisfy

$$\lim_{x\to\infty} \Pr(X > x) = C(\alpha)\, \gamma^{\alpha} \big((1+\beta)/2\big) x^{-\alpha} \tag{2}$$

and

$$\lim_{x\to-\infty} \Pr(X < x) = C(\alpha)\, \gamma^{\alpha} \big((1-\beta)/2\big) |x|^{-\alpha}\,, \tag{3}$$

i.e., both tails of the pdf of X are asymptotically Paretian, with tail shape parameter α.[5]

The function $C(\alpha)$ in Eqs. (2) and (3) is given by[6]

$$C(\alpha) = \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)} \quad \text{for } \alpha \neq 1 \tag{4}$$

and by $2/\pi$ for $\alpha = 1$.[7] The function $C(\alpha)$ is continuous and strictly decreasing over the interval (0, 2); furthermore, $\lim_{\alpha\downarrow 0} C(\alpha) = 1$ and $\lim_{\alpha\uparrow 2} C(\alpha) = 0$. In consequence, as $\alpha \uparrow 2$, proportionately less and less of the distribution's probability mass is located in its tail region. In addition, because the density's tails decline at an increasingly rapid rate as $\alpha \uparrow 2$, the likelihood of observing very large draws conditional on the draw coming from the tail region decreases as well. These observations explain why potentially very large sample sizes are required if one desires to estimate the index of stability with adequate precision if α is close to but smaller than 2.

Defining $\mathrm{E}\, |X|^{\xi} = \lim_{b\to\infty} \int_0^b x^{\xi}\, \mathrm{d}F_{|X|}(x)$, Eqs. (2) and (3) imply that $\mathrm{E}\, |X|^{\xi} < \infty$ for $\xi \in (0, \alpha)$ and $\mathrm{E}\, |X|^{\xi} = \infty$ for $\xi \geq \alpha$ for an α-stable random variable X with cumulative distribution function (cdf) F_X.[8] If $\alpha \in (1, 2)$—as is usually the case for empirical data

2 These three special cases are: the Gaussian distribution, $S(2, \gamma, 0, \delta) \equiv N(\delta, 2\gamma^2)$; the symmetric Cauchy distribution, $S(1, \gamma, 0, \delta)$; and the Lévy distribution, $S(0.5, \gamma, \pm 1, \delta)$; see, e.g., Zolotarev (1986, chapter 2) and Rachev et al. (2005, chapter 7).

3 Other parameterizations exist as well. See Nolan (2013) for a discussion of several alternative parameterizations.

4 $0 \cdot \ln 0$ is always interpreted as 0.

5 For $\alpha < 2$ and $\beta = +1\,(-1)$, i.e., for maximally right-skewed (left-skewed) distributions, only the right (left) tail is asymptotically Paretian. For $\alpha < 1$ and $\beta = +1$, $\Pr(X < \delta) = 0$, i.e., the distribution's support is bounded below by δ. Zolotarev (1986, Theorem 2.5.3) and Samorodnitsky and Taqqu (1994, pp. 17–18) provide expressions for the rate of decline of the non-Paretian tail if $\beta = \pm 1$ and $\alpha \geq 1$.

6 See Samorodnitsky and Taqqu (1994, p. 17).

7 The numerator and the second term in the denominator of Eq. (4) both converge to 0 as $\alpha \to 1$. The result $C(1) = 2/\pi$ is obtained by applying L'Hôpital's Rule.

8 Ibragimov and Linnik (1971, Theorem 2.6.4) show that this result holds not only for α-stable distributions but for *all* distributions in the domain of attraction of an α-stable distribution. Ibragimov and Linnik (1971, Theorem 2.6.1) provide necessary and sufficient conditions for a probability distribution to lie in the domain of attraction of an α-stable law.