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Statistical estimation of multivariate Ornstein–Uhlenbeck processes and applications to co-integration

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ABSTRACT

Ornstein–Uhlenbeck models are continuous-time processes which have broad applications in finance as, e.g., volatility processes in stochastic volatility models or spread models in spread options and pairs trading. The paper presents a least squares estimator for the model parameter in a multivariate Ornstein–Uhlenbeck model driven by a multivariate regularly varying Lévy process with infinite variance. We show that the estimator is consistent. Moreover, we derive its asymptotic behavior and test statistics. The results are compared to the finite variance case. For the proof we require some new results on multivariate regular variation of products of random vectors and central limit theorems. Furthermore, we embed this model in the setup of a co-integrated model in continuous time.

1. Introduction

In this paper we investigate the asymptotic properties of the least squares estimator for the model parameter in a multivariate Ornstein–Uhlenbeck model. Ornstein–Uhlenbeck processes are natural extensions of autoregressive processes of order one in discrete time to continuous time. Hence, they belong to the class of continuous-time autoregressive moving average (CARMA) processes. Not only that the Ornstein–Uhlenbeck process itself is a CARMA process but also the state space representation of a CARMA process leads to a multivariate Ornstein–Uhlenbeck process. Applications of CARMA processes include econometrics (see Bergstrom (1990), and Phillips (1974)), high-frequency financial econometrics (see Todorov (2009)), and financial mathematics (see Benth et al. (2010)). However, CARMA processes clearly have potential applications in all areas involving time series data, e.g. social sciences, medicine, biology or physics.

Typical stylized facts of high frequency financial time series as asset returns and exchange rates are jumps and a heavy tailed distribution which is peaked around zero. These characteristics were already noticed in the 60s by the influential works of Mandelbrot (1963) and Fama (1965). Thus, α-stable distributions as a generalization of a Gaussian distribution have often been discussed as more realistic models for asset returns than the usual normal distribution; see Rachev et al. (1999). More applications of heavy tailed distributions in economics and finance can be found in Adler et al. (1998), Rachev and Mittnik (2000) and Rachev (2003). Processes exhibiting infinite second variance have not only appeared in finance but also, e.g., in insurance, signal processing and teletraffic data. For an overview on the topic of heavy tailed distributions and their applications we refer to the excellent monograph of Resnick (2007). It is well known that for heavy tailed distributions standard statistical techniques do not apply in the usual way.

Further common features of high frequency financial time series are non-stationarity and time-varying volatility. A famous stochastic volatility model is the Ornstein–Uhlenbeck model propagated by Barndorff-Nielsen and Shephard (2001). They start from the classical Black–Scholes model for the log asset price

$$dX(t) = \left(\mu - \sigma^2\delta\right)\, dt + \sigma\, dB(t), \tag{1}$$

where $\mu, \delta \in \mathbb{R}$ are the instantaneous drift and the premium parameter, $\sigma > 0$ is the constant volatility and $(B(t))_{t\geq 0}$ is the Brownian motion, and plug in (1) the stochastic Ornstein–Uhlenbeck process $(\sigma^2(t))_{t\geq 0}$ as a volatility process instead of the constant volatility σ^2. This leads to

$$dX(t) = \left(\mu - \sigma^2(t)\delta\right)\, dt + \sigma(t)\, dB(t), \tag{2}$$

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where $(\sigma(t))_{t\geq 0}$ has the representation

$$\sigma^2(t) = \mathrm{e}^{-\lambda t}\sigma^2(0) + \mathrm{e}^{-\lambda t}\int_0^t \mathrm{e}^{\lambda s} L(ds) \quad \text{for } t \geq 0, \tag{3}$$

with $\lambda > 0$ and $(L(t))_{t\geq 0}$ a positive Lévy process also known as a subordinator. This model is capable of showing most of the stylized facts, e.g. volatility jumps, clustering and heavy tails (cf. Fasen et al. (2006)). It was used and studied in detail in finance, see e.g., Barndorff-Nielsen et al. (2002), Griffin and Steel (2006), and Roberts et al. (2004) and extended to the multivariate case with a multivariate Ornstein–Uhlenbeck type process by Pigorsch and Stelzer (2009).

Let $(\mathbf{L}(t))_{t\geq 0}$ be a p-dimensional Lévy process and $\Sigma \in \mathbb{R}^{d\times p}$, $\boldsymbol{\Lambda} \in \mathbb{R}^{d\times d}$, $d, p \in \mathbb{N}$, where the eigenvalues of $\boldsymbol{\Lambda}$ have strictly positive real parts. Then a multivariate Ornstein–Uhlenbeck process $(\mathbf{Z}(t))_{t\geq 0}$ in $\mathbb{R}^d$ is defined as

$$\mathbf{Z}(t) = \mathrm{e}^{-\boldsymbol{\Lambda} t}\mathbf{Z}(0) + \int_0^t \mathrm{e}^{-\boldsymbol{\Lambda}(t-s)}\,\Sigma\,\mathbf{L}(ds) \quad \text{for } t \geq 0. \tag{4}$$

Under our assumptions on $\mathbf{L}$, which will follow below, we can choose a stationary version of $(\mathbf{Z}(t))_{t\geq 0}$ since the eigenvalues of $\boldsymbol{\Lambda}$ have strictly positive real parts (see Sato and Yamazato (1984, Theorem 4.1)). For given observations

$$\mathbb{Z}_n' = (\mathbf{Z}(1), \ldots, \mathbf{Z}(n)) \in \mathbb{R}^{d\times n},$$

where we write $\mathbb{Z}_{i,j}' = (\mathbf{Z}(i), \ldots, \mathbf{Z}(j)) \in \mathbb{R}^{d\times(j-i+1)}$,

at a discrete-time grid, we study the properties of the least squares estimator for $\mathrm{e}^{-\boldsymbol{\Lambda}}$, since the estimation of $\boldsymbol{\Lambda}$ itself is not identifiable (cf. Lemma 3.1 below). An Ornstein–Uhlenbeck process observed at discrete time points is a multivariate AR(1) process with representation

$$\mathbf{Z}(k) = \mathrm{e}^{-\boldsymbol{\Lambda}}\mathbf{Z}(k-1) + \boldsymbol{\xi}_k$$

$$\text{where } \boldsymbol{\xi}_k = \int_{k-1}^k \mathrm{e}^{-\boldsymbol{\Lambda}(k-s)}\,\Sigma\,\mathbf{L}(ds) \text{ for } k \in \mathbb{N}. \tag{5}$$

The usual least squares estimator is then

$$\widehat{\mathrm{e}^{-\boldsymbol{\Lambda}_n}} = \mathbb{Z}_{2,n}'\mathbb{Z}_{1,n-1}(\mathbb{Z}_{1,n-1}'\mathbb{Z}_{1,n-1})^{-1}. \tag{6}$$

If the second moment of $\|\mathbf{L}(1)\|$ exists, then it is well known from the statistical inference of multivariate ARMA models that the estimator (6) is asymptotically normal and unbiased (see Hannan (1970, Chapter 6) or Proposition 3.2 below). In the finite variance case the least squares estimator is inefficient. More efficient estimators for the mean reversion parameter in one-dimensional Ornstein–Uhlenbeck models with finite variance are presented in Jongbloed et al. (2005), Brockwell et al. (2007), Taufer and Leonenko (2009) and Spiliopoulos (2009).

The main focus of this paper is to derive the asymptotic distribution of the least squares estimator (6), if $\|\mathbf{L}(1)\|$ has infinite variance. To be precise we study in detail the case where $\mathbf{L}(1)$ is multivariate regularly varying of an index less than 2, derive test statistics and compare our results to the case with finite second moments. The least squares estimator has the advantage that it is easy to implement and it performs much better in the heavy tailed model than in the finite variance case. In the limit the least squares estimator converges to a heavy tailed distribution which is a function of stable random variables. However, one cannot calculate the distribution analytically and it still depends on the unknown tail index of the underlying process. This makes it difficult to develop asymptotic approximations for the purpose of statistical inference.

Our result extends those of Davis and Resnick (1986) for the estimation of the autocorrelation function of a discrete-time AR(1) process from the one-dimensional case to the multivariate case. To the best of our knowledge statistical inference for multivariate linear processes with infinite variance has not been well explored in the literature yet, apart from the work of Davis et al. (1985) and Meerschaert and Scheffler (2000, 2001) regarding only the convergence in probability of the normalized autocovariance function and the cross-correlation. The estimation of heavy tailed continuous-time AR(1) models was only considered in Hu and Long (2007, 2009) paying attention only at one-dimensional stable Ornstein–Uhlenbeck processes. In contrast to ours, their observation grid gets finer if the time scale increases or they observe the process on a whole time interval.

It is worth noting that in the heavy tailed case of one-dimensional linear models in discrete time M-estimators, in particular, the least absolute deviation estimator, can be more efficient than least squares estimators because outliers do not dominate as in the case of least squares estimators (cf. Calder and Davis (1998), and Davis et al. (1992)). There, the least squares estimators give too much influence to outliers. Thus, the extension to M-estimators in the multivariate setup of continuous-time linear models will be considered in some future work and compared to the least squares estimator of this paper.

The second part of this paper is devoted to the application of Ornstein–Uhlenbeck processes and its statistical inference in the context of co-integration. Co-integration is a well known phenomenon in economic time series as e.g., interest rates on assets of different maturities, prices of commodities in different parts of the world, income and expenditure by the local government, the value of sales and production costs of an industry, and spot and future prices in commodity markets (see Engle and Granger (1991), Engle and White (1999), and Lütkepohl and Krätzig (2004)). This means that even though time series are non-stationary there exist linear combinations of them which render stationarity. Typical models for asset prices are exponential Lévy models (cf. Mandelbrot and Taylor (1967) and Eberlein (2009)). Although, exponential Lévy models are not able to capture stochastic volatility, they are the straightforward extension of the geometric Brownian motion in the Black–Scholes model modeling jumps and going away from the Gaussian assumption. The analytic form of the exponential Lévy model is simple, easier to handle and to fit to data than stochastic volatility models. In spread options and pairs trading, which is a popular investment strategy among hedge funds and investment banks, the concept is to find some pairs of assets which tend to move together in the long-run, i.e., they are co-integrated. There the logarithmic asset prices of two assets are modeled as

$$\begin{aligned} X(t) &= AY(t) + Z(t) \quad \text{for } t \geq 0, \\ Y(t) &= L(t) \quad \text{for } t \geq 0, \end{aligned} \tag{7}$$

where $(Z(t))_{t\geq 0}$ is a stationary Ornstein–Uhlenbeck process, $(L(t))_{t\geq 0}$ is a Lévy process and $A \in \mathbb{R}$ is a constant (see Benth and Benth (2006), Duan and Pliska (2004), Ekström et al. (2009), and Elliott et al. (2005)). Models of this type are also applied in electricity spot price dynamics (see Benth et al. (2008)). In the long-term the first asset behaves like a multiple of the second asset only in the short-term; there are some deviations modeled by $(Z(t))_{t\geq 0}$. The Ornstein–Uhlenbeck parameter λ of $(Z(t))_{t\geq 0}$ reflects the speed of mean reversion to the equilibrium and hence, this parameter is important to know and to estimate for the optimal strategy in a pairs trade. It is also possible to allow some short-term deviations of $(Y(t))_{t\geq 0}$ from $(L(t))_{t\geq 0}$ by adding a noise term (cf. Fasen (2012)). For ease of notation we neglect this here. The linear regression model (7) is commonly used and basic in econometrics. In a two-step procedure we will estimate A and $\mathrm{e}^{-\lambda}$. In this paper we investigate a multiple version of (7) and its statistical inference where the noise is modeled by a multivariate Ornstein–Uhlenbeck model.