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Constructing smooth tests without estimating the eigenpairs of the limiting process

Shih-Hsun Hsu [a], Chung-Ming Kuan [b,*]

[a] *Department of Economics, National Chengchi University, Taipei 116, Taiwan*
[b] *Department of Finance and CRETA, National Taiwan University, Taipei 106, Taiwan*

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ABSTRACT

Based on the well known Karhunen–Loève expansion, it can be shown that many omnibus tests lack power against "high frequency" alternatives. The smooth tests of Neyman (1937) may be employed to circumvent this power deficiency problem. Yet, such tests may be difficult to compute in many applications. In this paper, we propose a more operational approach to constructing smooth tests. This approach hinges on a Fourier representation of the postulated empirical process with known Fourier coefficients, and the proposed test is based on the normalized principal components associated with the covariance matrix of finitely many Fourier coefficients. The proposed test thus needs only standard principal component analysis that can be carried out using most econometric packages. We establish the asymptotic properties of the proposed test and consider two data-driven methods for determining the number of Fourier coefficients in the test statistic. Our simulations show that the proposed tests compare favorably with the conventional smooth tests in finite samples.

1. Introduction

Specification tests are indispensable tools in the process of model searching. There are basically two types of specification tests: directional tests and omnibus tests. A directional test focuses on certain alternatives of interest. While this test is powerful against the postulated alternatives, it is not a consistent test in general because it may not have power against some other alternatives. On the other hand, when researchers do not have any particular alternative in mind, they may prefer an omnibus test that is capable of detecting any potential deviations from the null hypothesis. There are numerous omnibus tests in the literature, such as the tests of martingale difference (e.g., Durlauf, 1991; Deo, 2000; Domínguez and Lobato, 2003) and general specification tests (e.g., Bierens, 1982, 1990; Bierens and Ploberger, 1997).

It can be verified that the limits of many omnibus tests are a functional of some stochastic (possibly Gaussian) process. By the well known Karhunen–Loève (KL) expansion (Karhunen, 1946; Loève, 1955), the limiting process in an omnibus test can be represented as a weighted sum of the products of the normalized principal components and eigenfunctions associated with the covariance operator, with the weights being the corresponding eigenvalues that diminish to zero. This suggests that such omnibus tests mainly have local power against a few orthogonal directions determined by the eigenfunctions with larger eigenvalues, but lack power against "high frequency" alternatives, i.e., the directions related to very small eigenvalues. See, e.g., Eubank and LaRiccia (1992), Bierens and Ploberger (1997), Janssen (2000), and Escanciano (2009) for more discussions.

The aforementioned power deficiency may be circumvented by employing "smooth" tests, in the sense of Neyman (1937); see also Eubank and LaRiccia (1992), Fan (1996), Ghosh and Bera (2001), and Escanciano and Mayoral (2010).[1] By construction, such smooth tests avoid diminishing weights in the limit and hence have more even power against a collection of directions. There have been many smooth tests in the literature, such as the tests of goodness of fit (Eubank and LaRiccia, 1992; Delgado and Stute, 2008), tests of martingale difference (Delgado et al., 2005; Escanciano and Mayoral, 2010), and general specification tests (Stute, 1997; Escanciano, 2009). A major difficulty of smooth tests is that they may not be easy to implement, because the statistics rely on the eigenpairs (eigenfunctions and corresponding eigenvalues) of the limiting process, which are usually unknown. It is, however, technically involved to estimate these eigenpairs; see, e.g., William and Seeger

* Corresponding author.
E-mail addresses: shhsu@nccu.edu.tw (S.-H. Hsu), ckuan@ntu.edu.tw (C.-M. Kuan).

[1] Another class of tests based on "kernel smoothing" also has better power against "high frequency" alternatives, e.g., Fan and Li (1996, 2000) and Fan (1998). In this paper, by smooth test we mean Neyman's smooth test.

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(2000, 2001), Carrasco et al. (2007), and Escanciano (2009). Thus, smooth tests are not readily available in many applications.

In this paper, we propose a more operational approach to constructing smooth tests. This approach hinges on a Fourier representation of the postulated empirical process with known Fourier coefficients. The proposed smooth test is based on the normalized principal components associated with the covariance matrix of finitely many Fourier coefficients. We thus need only a standard principal component analysis that can be carried out using most econometric and statistics packages. This is much simpler than estimating the eigenpairs of the limiting process. We establish the asymptotic properties of the proposed test and consider two data-driven methods for determining the number of Fourier coefficients in the test statistic. The first method, proposed by Inglot and Ledwina (2006), employs a model-selection criterion; the second method, studied in Inglot et al. (1994) and Fan (1996), is designed to maximize the asymptotic power. Monte Carlo simulations show that the proposed smooth tests compare favorably with the conventional smooth tests in finite samples.

This paper proceeds as follows. We review the conventional smooth test and propose a new smooth test in Section 2. The asymptotic properties of the proposed test and two data-driven tests are discussed in Section 3. Section 4 reports simulation results. Section 5 concludes the paper. All technical conditions and proofs are deferred to Appendix.

2. The proposed smooth test

2.1. The CvM and smooth tests

Many omnibus tests for model specification can be expressed in terms of a functional of an empirical process. In these tests, the behavior of the empirical process is essentially governed by its limiting process under the null hypothesis but tends to deviate from the limiting process otherwise; the chosen functional is then used to measure these deviations. The well known functionals include the Kolmogorov–Smirnov (KS) functional, i.e., the supremum functional, and Cramér–von Mises (CvM) functional, i.e., $\int f^2(s)ds$ for a square integrable function f.

Let X_n denote a square integrable empirical process on $[a, b]$ such that $X_n \Rightarrow X$ on $[a, b]$, where $\Rightarrow$ denotes weak convergence (of the associated probability measure) and X is also a square integrable process with zero mean. An omnibus test based on the CvM functional (hereafter the CvM test) is such that

$$\int_a^b X_n^2(\tau)d\tau \xrightarrow{d} \int_a^b X^2(\tau)d\tau,$$

where $\xrightarrow{d}$ stands for convergence in distribution. The covariance operator of X, $\mathbb{K}_X$, with the kernel $K_X(s,\tau) = \mathbb{E}[X(s)X(\tau)]$, is such that

$$\mathbb{K}_X f(\tau) := \int_a^b K_X(s,\tau)f(s)ds.$$

Corresponding to $\mathbb{K}_X$, there exist orthonormal eigenfunctions $\{\varepsilon_m(\cdot)\}$ and the associated eigenvalues $\{\alpha_m\}$ that satisfy

$$\int_a^b K_X(s,\tau)\varepsilon_m(s)ds = \alpha_m\varepsilon_m(\tau),$$

where $\alpha_1 \geq \alpha_2 \geq \cdots$.[2]

When X is quadratic mean continuous on $[a, b]$, its KL expansion is, in the quadratic mean sense,

$$\begin{aligned} X(\tau) &= \lim_{M\to\infty}\sum_{m=1}^{M} z_m\varepsilon_m(\tau) \\ &= \lim_{M\to\infty}\sum_{m=1}^{M}\sqrt{\alpha_m}z_m^*\varepsilon_m(\tau), \quad \tau\in[a,b], \end{aligned} \tag{1}$$

where $z_m = \int_b^a X(s)\varepsilon_m(s)ds$ are the principal components, which are mutually uncorrelated with variance α_m, and $z_m^* = z_m/\sqrt{\alpha_m}$ are the normalized principal components with variance one. It is readily seen that (1) is also a Fourier representation in the eigenfunctions $\{\varepsilon_m(\cdot)\}$, with z_m the Fourier coefficients. It follows from (1) and the Parseval Theorem that, in the quadratic mean sense, the limit of the CvM test is:

$$\int_a^b X^2(\tau)d\tau = \lim_{M\to\infty}\sum_{m=1}^{M} z_m^2 = \lim_{M\to\infty}\sum_{m=1}^{M}\alpha_m(z_m^*)^2. \tag{2}$$

When $\mathbb{K}_X$ is square integrable, $\alpha_m \to 0$ as m tends to infinity.[3] Therefore, the CvM test based on X_n virtually has no local power against "high frequency" alternatives, i.e., the directions corresponding to very small eigenvalues (i.e., α_m with large m).

To alleviate the power deficiency in the CvM test, it is natural to construct a test whose limit does not involve the diminishing weights α_m. To this end, consider the process $\Xi_M(\tau) = \sum_{m=1}^{M} z_m^*\varepsilon_m(\tau)$ and note that

$$\int_a^b \Xi_M^2(\tau)d\tau = \sum_{m=1}^{M}(z_m^*)^2, \tag{3}$$

cf. (2). Letting $\hat{z}_{m,n}^*$ be consistent estimates of z_m^* based on the sample of size n, we may construct the following test: for a given M,

$$T_{n,M} = \sum_{m=1}^{M}(\hat{z}_{m,n}^*)^2. \tag{4}$$

It is clear that the limit of $T_{n,M}$ is (3). This is a smooth test in the sense of Neyman (1937); see also Ghosh and Bera (2001) for a review of Neyman's smooth test. When X is a Gaussian process, it is well known that z_m are independent Gaussian random variables so that z_m^* are i.i.d. $\mathcal{N}(0, 1)$. In this case, (3) has a $\chi^2(M)$ distribution.

Compared with the CvM test with the limit (2), the smooth test $T_{n,M}$ ought to have more even power against the directions corresponding to the first M principal components. For the remaining directions corresponding to other components (z_m^* with $m > M$), $T_{n,M}$ would have no power. Yet, the power loss may be minimal because the CvM test itself has little power against these directions, due to the presence of the diminishing weights α_m. On the other hand, $T_{n,M}$ cannot be easily implemented unless the eigenpairs of the covariance operator $\mathbb{K}_X$, hence the normalized principal components z_m^*, are known. Unfortunately, except for some special X processes, such as the standard Wiener process and Brownian bridge, the eigenpairs are unknown and need to be estimated. Estimating the eigenpairs of the covariance operator is, however, technically involved; see, e.g., William and Seeger (2000, 2001), Carrasco et al. (2007), and Escanciano (2009). Therefore, constructing smooth tests may not be as straightforward as one would think.

[2] The orthonormal eigenfunctions $\{\varepsilon_m(\cdot)\}$ satisfy $\int_a^b e_m(s)e_n(s)ds = 0$ and $\int_a^b e_m^2(s)ds = 1$.

[3] By square integrability of $\mathbb{K}_X$,

$$\int_a^b\int_a^b K_X^2(s,\tau)d\tau ds = \int_a^b\left(\sum_{m=1}^{\infty}\alpha_m^2\varepsilon_m^2(s)\right)ds = \sum_{m=1}^{\infty}\alpha_m^2 < \infty,$$

where the first equality follows from the Parseval's theorem. It follows that $\alpha_m \to 0$ as $m \to \infty$. It can also be shown that $\alpha_m \to 0$ when $\mathbb{K}_X$ is integrable.