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Estimation of long-run parameters in unbalanced cointegration

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ABSTRACT

This paper analyses the asymptotic properties of nonlinear least squares estimators of the long run parameters in a bivariate unbalanced cointegration framework. Unbalanced cointegration refers to the situation where the integration orders of the observables are different, but their corresponding balanced versions (with equal integration orders after filtering) are cointegrated in the usual sense. Within this setting, the long run linkage between the observables is driven by both the cointegrating parameter and the difference between the integration orders of the observables, which we consider to be unknown. Our results reveal three noticeable features. First, superconsistent (faster than $\sqrt{n}$-consistent) estimators of the difference between memory parameters are achievable. Next, the joint limiting distribution of the estimators of both parameters is singular, and, finally, a modified version of the "Type II" fractional Brownian motion arises in the limiting theory. A Monte Carlo experiment and the discussion of an economic example are included.

1. Introduction

Since the seminal paper of Engle and Granger (1987), cointegration, which has traditionally focused on the case of unit root observables with weak dependent cointegrating errors, has been a fertile field of research. This original idea has been generalized in various directions. Among these, one of the main developments is that of fractional cointegration, which given the concept of fractional integration (introduced by Granger and Joyeux, 1980), extends and encompasses naturally the standard notion of cointegration. In the simple bivariate case two processes sharing the same integration order (say δ) are cointegrated if there is a linear combination of them with integration order smaller than δ. In a multivariate situation several definitions are available (see, e.g., Robinson and Yajima, 2002), although all of them share the idea of reducing-order linear combination. Inference procedures for fractional cointegration have been developed by, e.g., Jeganathan (1999), Robinson and Marinucci (2001), Robinson and Yajima (2002), Robinson and Hualde (2003), Marmol and Velasco (2004), Christensen and Nielsen (2006), Hualde and Robinson (2007, 2010), Nielsen and Frederiksen (2011), Johansen and Nielsen (2012). However, most of the previous studies have not captured the situation termed by Hualde (2006) as unbalanced cointegration (UC hereinafter), with the important exceptions of Johansen (2008) and Franchi (2010), which give conditions under which such situation might arise in a fractional vector autoregressive model, but do not develop inferential procedures. In the simple bivariate case, UC denotes a situation where the integration orders of the observables are different, but their corresponding balanced versions (where one of the series is filtered adequately so it has identical integration order to the other one) are cointegrated in the usual sense. This can be seen as a particular case of the so-called polynomial cointegration, which in the integer order case has been studied by, e.g., Johansen (1995).

Denoting by θ the imbalance between the integration orders of the two observables, Hualde (2006) discusses two situations, one where $\theta = \theta_n \to 0$ as $n \to \infty$, (n denoting sample size), named weak UC, and the other where θ is an unknown fixed real number different from zero, named strong UC. While the former situation is treated with a good deal of theoretical rigour, the latter (denoted simply as UC hereinafter) is just briefly discussed. UC poses interesting challenges, meanly because while in a "balanced" bivariate situation (where $\theta = 0$), if there exists cointegration, the cointegrating parameter drives the long run linkage between the observables, if there is UC (so $\theta \neq 0$), it is both θ and the cointegrating parameter which are relevant in order to explain the long run comovements of the observables. Thus, from a theoretical viewpoint, allowing for the possibility of an unknown (and possibly nonzero) θ is relevant, especially noting that misspecification of θ could have very distorting effects (see Hualde, 2006). In addition, empirical researchers usually admit the possibility that $\theta = 0$ as the outcome of testing procedures (e.g. Dickey and Fuller, 1979, or Robinson and Yajima's, 2002, test for equality of orders), so, even if $\theta = 0$, a safer option is to take the agnostic approach of considering θ to be an

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unknown parameter, and not imposing knowledge of it in the estimation procedure.

While the main focus of the paper is to present formal theoretical discussion of the limiting properties of particular estimators in UC, we would also like to motivate UC from an empirical perspective. Interestingly, we find that UC relates directly to the idea of multicointegration proposed by Granger and Lee (1989, 1990) (which can be also seen as a particular case of polynomial cointegration). The idea of multicointegration appears to be the most empirically relevant situation involving cointegration between processes with different but known integer orders of integration and, as stated by Engsted and Haldrup (1999), this phenomenon is likely to occur in stock-flow models. Here, two flow variables (usually characterized as unit roots) cointegrate in the standard way, and the cumulated cointegrating error (stock variable) cointegrates with at least one of the flow observables. In a seminal contribution, Granger and Lee (1989) applied this idea to the relationship between production and sales (flow variables) in a given industry, exploring also the possibility of cointegration between the stock of inventories (accumulated change of inventory) and sales, which would support the idea of targeting (the target level of inventory being just a fixed proportion of sales). An alternative analysis of the relationship between inventories and sales was performed by Banerjee and Mizen (2006). Other works explore the existence of multicointegration between housing starts, completions (flow variables) and housing units under construction (stock) (see Lee, 1992), government spending, revenues and debt (Leachman, 1996; Leachman and Francis, 2002; Leachman et al., 2005), imports, exports and external debt (Leachman and Francis, 2000, 2002), or real per capita private consumption expenditure, real per capita disposable income and stock of consumer's wealth (Siliverstovs, 2006).

The role of UC within the framework of multicointegration can be explained as follows. One of the key assumptions behind the idea of multicointegration is that the stock variable (accumulated cointegrating error) must have the same integration order as that of the flows. This necessarily implies that the cointegrating gap (that is the reduction in order due to the cointegrating relation) in the relationship between the flows be equal to one. However, empirical works in fractional cointegration show substantial evidence in favour of smaller cointegrating gaps (see, e.g., Gil-Alana and Hualde, 2009), which in terms of multicointegration means that the stock variable would have a larger integration order than that of the flows. Particularizing this possibility, e.g., to the relationship between inventories (h_t) and sales (s_t), even admitting the possibility that s_t is a unit root, if the cointegrating gap arising from the relationship between production and sales is d, the only interesting cointegrating possibility between inventories and sales would be that between h_t and $\Delta^{\theta} s_t$, where $\Delta = 1 - L$, L being the lag operator, and $\theta = d - 1$, noting that h_t and $\Delta^{\theta} s_t$ would share the same integration order $2 - d$ (a proper definition of the fractional operator Δ^{θ} will be given below). If $d \neq 1$, this would exemplify the situation of UC. Given that $\Delta^{\theta} s_t$ is a linear combination of present and past values of s_t, UC would lead to the idea of dynamic targeting, where the target level of inventories is a proportion of present and past sales.

Multicointegration is not the only setting where the idea of UC might be useful. Another motivating example is that of predictive regressions, where rates of return are regressed against the lagged values of a explanatory variable (see, e.g., Torous et al., 2004). Here, it is standard to consider the rates of return as weak dependent, whereas the assumption of weak dependence for the regressor is usually unsatisfactory. We can exemplify this situation by the forward premium anomaly, which consists on surprising negative estimates from the regression of the change in the logarithms of the spot exchange rate (considered to be weak dependent) on the forward premium (stationary long memory or nonstationary but mean reverting), where the theory predicts a value of one for that slope (see e.g. Bekaert, 1996; Bekaert et al., 1997). Baillie and Bollerslev (2000) refer to the forward premium anomaly as a statistical problem caused by the different integration orders of dependent and explanatory variables, and Maynard and Phillips (2001) gave theoretical justification to this phenomenon. Recently, Maynard et al. (2013) provided an interesting empirical analysis which, in particular, takes into account the possible imbalance between the memories of the dependent variable and regressor. As will be seen below (Remark 8), the results we obtain in the present paper are not directly applicable to their problem, but an alternative approach focused on modelling the relation between the spot exchange rate (possibly unit root) and the integrated forward premium (which could have memory larger than one) might fall within the UC setting. In any case, even if the situation considered is not characterized by UC, the techniques developed in the present paper can be very useful when dealing with cases, like that of Maynard et al. (2013), where there is imbalance between the integration orders of dependent variable and regressor.

The rest of the paper is organized as follows. In Section 2 we present a model of UC and estimators of the relevant parameters, justifying also their limiting properties. A Monte Carlo experiment of finite sample performance is presented in Section 3. An empirical example is discussed in Section 4 and, finally, we conclude in Section 5.

2. Model and estimation of long-run parameters

Before introducing our proposed model we present some definitions. We say that a vector process ζ_t is integrated of order zero ($I(0)$) if $\zeta_t - E(\zeta_t)$ is covariance stationary with spectral density finite and nonsingular at all frequencies. Then, denoting by r_{it} the ith element of an arbitrary vector r_t, we say, as in Robinson and Gerolimetto (2006), that a scalar process ξ_t is integrated of order d ($I(d)$) if for any $l \times 1$ zero mean $I(0)$ vector ζ_t, $\xi_t - E(\xi_t) = \sum_{k=1}^{l} \zeta_{kt}(-d_k)$, with $d = \max_{1 \le k \le l} d_k$, where for a scalar or vector process ξ_t and real number α,

$$\xi_t(\alpha) = \Delta^{\alpha}\{\xi_t 1(t > 0)\} = \sum_{j=0}^{t-1} a_j(-\alpha)\,\xi_{t-j},$$
$$a_j(\alpha) = \frac{\Gamma(j+\alpha)}{\Gamma(\alpha)\Gamma(j+1)}, \quad \alpha \neq 0, -1, \ldots, \tag{1}$$

where $1(\cdot)$ denotes the indicator function (so $\xi_t 1(t > 0) = \xi_t$ if $t > 0$; $= 0$ if $t \le 0$), $\Gamma(\cdot)$ represents the gamma function, taking $\Gamma(\alpha) = \infty$ for $\alpha = 0, -1, -2, \ldots$, and $\Gamma(0)/\Gamma(0) = 1$. Note that introducing the indicator function in (1) leads to a truncation: in particular, $\zeta_{kt}(-d_k) = \sum_{j=0}^{t-1} a_j(d_k)\,\zeta_{k,t-j}$, which can be compared to the untruncated sum $\Delta^{-d_k}\zeta_{kt} = \sum_{j=0}^{\infty} a_j(d_k)\,\zeta_{k,t-j}$. The reason why the indicator is introduced here is that it ensures that processes are well defined in mean square sense. In fact, $\Delta^{-d_k}\zeta_{kt}$ is well defined in mean square sense just if $d_k < 1/2$, whereas $\zeta_{kt}(-d_k)$ is well defined for any value of d_k. Thus, the truncation allows a uniform treatment of all integration orders, although, related to the previous expression, it is certainly unnecessary when $d_k < 1/2$, in which case $\Delta^{-d_k}\zeta_{kt}$ is stationary. This truncation is very standard in the fractional integration and cointegration literature and originates the so-called Type II fractional processes. Additionally, we say that two scalar processes sharing the same integration order are cointegrated if a linear combination of them has a smaller integration order.

We introduce a bivariate model of UC. Let y_t, x_t, $t \in Z$, $Z = \{t : t = 0, \pm 1, \ldots\}$, be two scalar observable series generated by model

$$y_t = \mu + \nu x_t(\theta) + u_{1t}(-\gamma), \tag{2}$$

$$x_t = u_{2t}(-(\delta + \theta)). \tag{3}$$