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Density approximations for multivariate affine jump-diffusion processes

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ABSTRACT

We introduce closed-form transition density expansions for multivariate affine jump-diffusion processes. The expansions rely on a general approximation theory which we develop in weighted Hilbert spaces for random variables which possess all polynomial moments. We establish parametric conditions which guarantee existence and differentiability of transition densities of affine models and show how they naturally fit into the approximation framework. Empirical applications in option pricing, credit risk, and likelihood inference highlight the usefulness of our expansions. The approximations are extremely fast to evaluate, and they perform very accurately and numerically stable.

1. Introduction

Most observed phenomena in financial markets are inherently multivariate: stochastic trends, stochastic volatility, and the leverage effect in equity markets are well-known examples. The theory of affine processes provides multivariate stochastic models with a well established theoretical basis and sufficient degree of tractability to model such empirical attributes. They enjoy much attention and are widely used in practice and academia. Among their best-known proponents are Vasicek's interest rate model (Vasicek, 1977), the square-root model (Cox et al., 1985), Heston's model (cf. Heston, 1993), and affine term structure models (Duffie and Kan, 1996; Dai and Singleton, 2000; Collin-Dufresne et al., 2008). Affine models owe their popularity and their name to their key defining property: their characteristic function is of exponential affine form and can be computed by solving a system of generalized Riccati differential equations (cf. Duffie et al. (2003)). This allows for computing transition densities and transition probabilities by means of Fourier inversion (Duffie et al., 2000). Transition densities constitute the likelihood which is an ingredient for both frequentist and Bayesian econometric methodologies.[1] Also, they appear in the pricing of financial derivatives. However, Fourier inversion is a very delicate task. Complexity and numerical difficulties increase with the dimensionality of the process. Efficient density approximations avoiding the need for Fourier inversion are therefore desirable.

This paper is concerned with directly approximating the transition density without resorting to Fourier inversion techniques. We pursue a polynomial expansion approach, an idea that has been proposed by Wong and Thomas (1962), Wong (1964), Schoutens (2000), Schaumburg (2001), Aït-Sahalia (2002), and Hurn et al.

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[1] Various other approaches for parameter estimation for discretely observed Markov processes can be found in the literature (excellent comprehensive surveys are for example in Hurn et al., 2007; Sørensen, 2004; Aït-Sahalia, 2007). The approaches range from likelihood approximation using Bayesian data augmentation (Roberts and Stramer, 2001; Elerian et al., 2001; Eraker, 2001; Jones, 1998), estimating functions (Bibby et al., 2004), up to the efficient method of moment (Gallant and Tauchen, 2009). Only few of them make use of the properties of affine models, however (e.g. Singleton, 2001; Bates, 2006).

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(2008) among others for univariate diffusion processes. Extensions for multivariate (jump-)diffusions do exist in Aït-Sahalia (2008) (with applications in Aït-Sahalia and Kimmel, 2007, 2010), and Yu (2007), but they follow a different route by approximating the Kolmogorov forward-, and backward partial differential equations. Our approach exploits a crucial property of affine processes. Under some technical conditions, conditional moments of all orders exist and can be explicitly computed in closed form as the solution of a matrix exponential. This ensures that the coefficients of the polynomial expansions can be computed without an approximation error.

We present a general theory of density approximations with several traits of the affine model class in mind. The assumptions made for the general theory are then justified by proving existence and differentiability of the true, unknown transition densities of affine models. These theoretical results, contrary to the density approximations themselves, do rely on Fourier theory. Specifically, we investigate the asymptotic behavior of the characteristic function with novel ODE techniques.

Specializing to affine models we improve earlier work along several lines. Our method (i) is applicable to multivariate models; (ii) works equally well for reducible and irreducible processes in the sense of Aït-Sahalia (2008),[2] in particular stochastic volatility models; (iii) produces density approximations the quality of which is independent of the time interval between observations; (iv) allows for expansions on the "correct" state space. That is, the support of the density approximation agrees with the support of the true, unknown transition density as in Hurn et al. (2008) and Schoutens (2000); (v) produces density approximations that integrate to unity by construction, hence are much more amenable to applications that demand the constant of proportionality than the purely polynomial expansions from Aït-Sahalia (2008).[3] This includes Wishart processes (Bru, 1991) and even general affine matrix-valued processes (Cuchiero et al., 2010). This paper therefore provides a unified framework for econometric inference for financial models, because in applications one typically needs to evaluate, both, the transition densities themselves, as well as integrals of payoff functions against the transition densities for model-based asset pricing. This complements the methods recently developed in Chen and Joslin (2011) and Kristensen and Mele (2011), which are aimed at asset pricing.[4]

The paper proceeds as follows: Section 2 develops a general theory of orthonormal polynomial density approximations in certain weighted $\mathcal{L}^2$ spaces under suitable integrability and regularity assumptions. These may be validated by the sufficient criteria presented subsequently in Section 3. The density approximations are then specialized within the context of affine processes: Section 4 reviews the affine transform formula and the polynomial moment formula for affine processes, which in turn allows the aforementioned polynomial approximations. The main theoretical contribution – general results on existence and differentiability of transition densities of affine processes – is elaborated in Section 4.3. In Section 5 we introduce candidate weight functions and the Gram–Schmidt algorithm to compute orthonormal polynomial bases corresponding to these weights, along with important examples. Section 6 details computation of the ingredients to the density expansions using an explicit example. Section 7 relates existing techniques for density approximations to ours. An empirical study is presented in Section 8: Applications in stochastic volatility (Section 8.1), option pricing (Section 8.2), credit risk (Section 8.3), and likelihood inference (Section 8.4), support the tractability and usefulness of the likelihood expansions. Section 9 concludes. The proofs of our main results are given in an Appendix.

In the paper we will use the following notational conventions. The nonnegative integers are denoted by $\mathbb{N}_0$. The length of a multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ is defined by $|\alpha| = \alpha_1 + \cdots + \alpha_d$, and we write $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d}$ for any $\xi \in \mathbb{R}^d$. The degree of a polynomial $p(x) = \sum_{|\alpha|\geq 0} p_\alpha x^\alpha$ in $x \in \mathbb{R}^d$ is defined as $\deg p(x) = \max\{|\alpha| \mid p_\alpha \neq 0\}$. For the likelihood ratio functions below we define $0/0 = 0$. The class of p-times continuously differentiable (or continuous, if $p = 0$) functions on $\mathbb{R}^d$ is denoted by C^p.

2. Density approximations

Let g denote a probability density on $\mathbb{R}^d$ whose polynomial moments

$$\mu_\alpha = \int_{\mathbb{R}^d} \xi^\alpha \, g(\xi)\, d\xi$$

of every order $\alpha \in \mathbb{N}_0^d$ exist and are known in closed form. For example, g may denote the pricing density in a financial market model. Typically, g is not known in explicit form, and needs to be approximated. Let w be an auxiliary probability density function on $\mathbb{R}^d$. The aim is to expand the likelihood ratio g/w in terms of orthonormal polynomials of w in order to get an explicit approximation of the unknown density function g. This can be formalized as follows. Define the weighted Hilbert space $\mathcal{L}^2_w$ as the set of (equivalence classes of) measurable functions f on $\mathbb{R}^d$ with finite $\mathcal{L}^2_w$-norm defined by

$$\|f\|^2_{\mathcal{L}^2_w} = \int_{\mathbb{R}^d} |f(\xi)|^2 \, w(\xi)\, d\xi < \infty.$$

Accordingly, the scalar product on $\mathcal{L}^2_w$ is denoted by

$$\langle f, h\rangle_{\mathcal{L}^2_w} = \int_{\mathbb{R}^d} f(\xi)\, h(\xi)\, w(\xi)\, d\xi.$$

We will now proceed under the following assumptions. Sufficient conditions for the assumptions to hold are provided in Section 3.

Assumption 1. There exists an orthonormal basis of polynomials $\{H_\alpha \mid \alpha \in \mathbb{N}_0^d\}$ of $\mathcal{L}^2_w$ with $\deg H_\alpha = |\alpha|$. This implies $H_0 = 1$ in particular.

Assumption 2. The likelihood ratio function g/w lies in $\mathcal{L}^2_w$. This is equivalent to $\int_{\mathbb{R}^d} \frac{g(\xi)^2}{w(\xi)}\, d\xi < \infty$.

Consequently, the coefficients

$$c_\alpha = \left\langle \frac{g}{w}, H_\alpha \right\rangle_{\mathcal{L}^2_w} = \int_{\mathbb{R}^d} H_\alpha(\xi)\, g(\xi)\, d\xi \quad (=1 \text{ for } \alpha = 0)$$

are well-defined and given explicitly[5] in terms of the coefficients of H_α and the polynomial moments μ_α of g. Moreover, according to standard $\mathcal{L}^2_w$-theory, the sequence of pseudo-likelihood ratios[6]

2 A model is said to be *reducible* in the sense of Aït-Sahalia (2008) if its diffusion function can be transformed one-to-one into a constant. Otherwise it is termed *irreducible*.

3 The Markov chain Monte Carlo sampling schemes from Stramer et al. (2010) accommodate Bayesian likelihood-based inference using expansions from Aït-Sahalia (2008) even in absence of the normalizing constant, but at a high computational cost.

4 It is of course conceivable to mix the mentioned methods. For example, one could use transition densities developed in this paper, while approximating asset prices using the generalized Fourier transform in Chen and Joslin (2011), whenever the payoff function allows it, or the error expansion method from Kristensen and Mele (2011).

5 This is an advantage over the method in Aït-Sahalia (2002) which also relies on series expansions, where the coefficients are functions of expectations of nonlinear moments, and therefore have to be approximated in general.

6 See Footnote 7 for an explanation of this terminology.