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Estimation in threshold autoregressive models with a stationary and a unit root regime

Jiti Gao [a,b], Dag Tjøstheim [c,*], Jiying Yin [a,b]

[a] School of Economics, University of Adelaide, Adelaide SA 5005, Australia
[b] Department of Econometrics and Business Statistics, Monash University, Caulfield East VIC 3145, Australia
[c] Department of Mathematics, University of Bergen, Bergen 5008, Norway

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ABSTRACT

This paper treats estimation in a class of new nonlinear threshold autoregressive models with both a stationary and a unit root regime. Existing literature on nonstationary threshold models has basically focused on models where the nonstationarity can be removed by differencing and/or where the threshold variable is stationary. This is not the case for the process we consider, and nonstandard estimation problems are the result.

This paper proposes a parameter estimation method for such nonlinear threshold autoregressive models using the theory of null recurrent Markov chains. Under certain assumptions, we show that the ordinary least squares (OLS) estimators of the parameters involved are asymptotically consistent. Furthermore, it can be shown that the OLS estimator of the coefficient parameter involved in the stationary regime can still be asymptotically normal while the OLS estimator of the coefficient parameter involved in the nonstationary regime has a nonstandard asymptotic distribution. In the limit, the rate of convergence in the stationary regime is asymptotically proportional to $n^{-\frac{1}{4}}$, whereas it is n^{-1} in the nonstationary regime. The proposed theory and estimation method are illustrated by both simulated data and a real data example.

1. Introduction

Ordinary unit root models have just one regime, whereas ordinary threshold models have several regimes, but are stationary. In this paper, we study a threshold model that has unit-root behavior in one regime and acts as a stationary process in another regime. More specifically, we consider a parametric threshold autoregressive (TAR) model of the form

$$y_t = \alpha_1 y_{t-1} I[y_{t-1} \in C_\tau] + \alpha_2 y_{t-1} I[y_{t-1} \in D_\tau] + e_t, \quad 1 \le t \le n, \tag{1.1}$$

where C_τ is a subset of $R^1 = (-\infty, \infty)$ indexed by $\tau > 0$, $D_\tau = C_\tau^c = R^1 - C_\tau$ is the complement of C_τ, τ is essentially assumed to be known in the asymptotic analysis in this paper, $-\infty < \alpha_1, \alpha_2 < \infty$ are assumed to be unknown parameters, but will be estimated under the assumption that $\alpha_2 = 1$, the distribution of $\{e_t\}$ is absolutely continuous with respect to Lebesgue measure with $p_e(\cdot)$ being the density function satisfying $\inf_{x \in C} p_e(x) > 0$ for all compact sets C, $\{e_t\}$ is assumed to be a sequence of independent and identically distributed (i.i.d.) random errors with $E[e_1] = 0$, $0 < \sigma^2 = E[e_1^2] < \infty$ and $E[e_1^4] < \infty$, $\{e_t\}$ and $\{y_s\}$ are assumed to be mutually independent for all $s < t$, and n is the sample size of the time series. Let $y_0 = 0$ throughout this paper. Even though (1.1) is the simplest possible of the type of models we are discussing, it requires nonstandard techniques using the theory of null recurrent Markov chains. A few results of this theory are reviewed in Appendix A.

The vast majority of threshold models used have been stationary models, i.e., models for which $|\alpha_1| < 1$ and $|\alpha_2| < 1$ in the first-order case. Such models were introduced by Tong and Lim (1980). See also Tong (1983, 1990). Among later contributions, Chan (1990, 1993) consider both estimation and testing problems for the case where $\{y_t\}$ of (1.1) is stationary. His work is extended in Li and Ling (2011). Pham et al. (1991) consider a nonlinear unit-root problem and establish strong consistency results for the ordinary least squares (OLS) estimators of α_1 and α_2 for the case where (α_1, α_2) lie on the boundary, Hansen (1996) rigorously establishes an asymptotic theory for the likelihood ratio test for a threshold, Chan and Tsay (1998) discuss a related continuous-time TAR model, and Hansen (2000) proposes a new approach to estimating stationary TAR models. More recently, Liu et al. (2011) extend the discussion of Pham et al. (1991) by establishing an asymptotic distribution of the OLS estimator of α_2 for the case

* Corresponding author.
E-mail address: dag.tjostheim@math.uib.no (D. Tjøstheim).

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where $C_\tau = (-\infty, \tau]$ and either $\alpha_2 = 1$ and $\alpha_1 < 1$ or $\alpha_2 > 1$ and $\alpha_1 \le 1$ hold.

There have been other extensions to the nonstationary case, see in particular Caner and Hansen (2001), thus having a class of models that allows for both nonlinearity and nonstationarity, and where these properties can be (Caner and Hansen, 2001) separately tested for. The nonstationarity of these models under the null hypothesis has been of a rather restricted form, thus typically regarding both $y_t - y_{t-1}$ and the threshold variable to be stationary. In the first-order case, this leads to a somewhat degenerate model that under the null hypothesis has $H_0: \alpha_1 = \alpha_2 = 1$ in

$$y_t - y_{t-1} = (\alpha_1 - 1)\, y_{t-1} I[z_t \in C_\tau] + (\alpha_2 - 1)\, y_{t-1} I[z_t \in D_\tau] + e_t, \tag{1.2}$$

where $\{z_t\}$ is a sequence of stationary threshold variables, $C_\tau = (-\infty, \tau]$ and $D_\tau = (\tau, \infty)$. The parameters α_1 and α_2 can then be estimated under H_0, which leads to a pure random walk model for (1.2) but more general difference type models for the higher-order case are treated in Caner and Hansen (2001). The authors also point out that there are several nonstationary alternatives when H_0 does not hold. One of the alternatives to H_0 is as follows:

$$H_1: |\alpha_1| \ne 1 \quad \text{and} \quad \alpha_2 = 1, \tag{1.3}$$

which does not imply $y_t - y_{t-1}$ is stationary under H_1. For more references, including econometric interpretations of threshold effects, we refer to Teräsvirta et al. (2010); see in particular Sections 3.2.2, 8.2.3, 11.5 and 11.8.

We allow for more general forms of nonstationarity in which we do not require $y_t - y_{t-1}$ to be stationary, nor do we require the threshold variable to be stationary. To the best of our knowledge, estimation in this situation has not been treated before in the literature. In the present paper, for simplicity, we only treat the first-order case, but the theory can be extended to higher-order and vector models, making it possible to introduce threshold cointegration models in this context. It is also possible to allow nonlinear behavior in the regime C_τ. This is done by replacing the linear function $\alpha_1 y$ by a nonparametric function, also implicitly including an intercept in the model.

Although our focus in this paper is to estimate α_1 and α_2 and then study asymptotic properties of the proposed estimates in Section 2.1 when τ is assumed to be known, we propose an estimation procedure for the τ parameter in Section 2.2 when τ is unknown. Since the case of both $|\alpha_1| < 1$ and $|\alpha_2| < 1$ and the case of $\alpha_1 = \alpha_2 = 1$ have already been discussed in the literature (Chan, 1993; Hansen, 2000), we are interested in proposing an estimation method to deal with model (1.1) where C_τ is either a compact subset of R^1 or a set of type $(-\infty, \tau]$ or $[\tau, \infty)$, and where $\alpha_2 = 1$ and α_1 may be larger or smaller than one in absolute value. Model (1.1) may be used to detect and then estimate structural change from one regime to another. Note that τ can be a vector of unknown parameters. In the case where $C_\tau = [\tau_1, \tau_2]$ with $-\infty < \tau_1 < \tau_2 < \infty$, $\tau = (\tau_1, \tau_2)$. It is shown in Section 2 that the OLS estimator of α_1 is asymptotically consistent with a rate of convergence which in the limit is proportional to $n^{-\frac{1}{4}}$ where we can even let $|\alpha_1| > 1$ when C_τ is compact. By contrast, the OLS estimator of α_2 is asymptotically consistent with the super n-rate of convergence. In a related paper by Liu et al. (2011), the authors have established similar results for $\widehat{\alpha}_2$, but have not established any asymptotic theory for $\widehat{\alpha}_1$.

The organization of this paper is as follows. Section 2 establishes asymptotic distributions of the OLS estimators of α_1 and α_2 and contains an estimation procedure for the threshold parameter τ. Section 3 discusses an extension of model (1.1) to a semiparametric threshold autoregressive (SEMI-TAR) model. Examples of implementation are given in Section 4. The paper concludes in Section 5.

We will use the theory of β-null recurrent Markov chains in this paper and some general results about these processes are given in Appendix A. Much more details can be found in Karlsen and Tjøstheim (2001), hereafter referred to as KT. The theory of the present paper is different from the theory of KT in several aspects. In contrast to KT, we consider a parametric nonstationary model. The absence of a kernel function makes it harder to prove existence of moments. On the other hand, the autoregressive structure makes it difficult to apply the local-time regression technique of Park and Phillips (2001) and Wang and Phillips (2009a,b). The threshold structure and the splitting into two regimes are what makes it possible to employ some of the theory of KT in the present situation. The mathematical proofs of our theory are given in Appendix B.

2. Estimation in parametric threshold autoregressive models

We propose an ordinary least squares (OLS) estimation method for the unknown parameters α_1 and α_2 in Section 2.1. Some remarks about estimation of the τ parameter are given in Section 2.2.

2.1. OLS estimation method and asymptotic theory

Consider model (1.1). It is obvious that α_1 and α_2 can be estimated by the ordinary least squares estimators

$$\widehat{\alpha}_1 = \widehat{\alpha}_1(\tau) = \frac{\sum_{t=1}^n y_t\, y_{t-1} I[y_{t-1} \in C_\tau]}{\sum_{t=1}^n y_{t-1}^2 I[y_{t-1} \in C_\tau]} \quad \text{and} \tag{2.1}$$

$$\widehat{\alpha}_2 = \widehat{\alpha}_2(\tau) = \frac{\sum_{t=1}^n y_t\, y_{t-1} I[y_{t-1} \in D_\tau]}{\sum_{t=1}^n y_{t-1}^2 I[y_{t-1} \in D_\tau]}. \tag{2.2}$$

This implies that

$$\widehat{\alpha}_1 - \alpha_1 = \frac{\sum_{t=1}^n e_t\, y_{t-1} I[y_{t-1} \in C_\tau]}{\sum_{t=1}^n y_{t-1}^2 I[y_{t-1} \in C_\tau]} \quad \text{and} \tag{2.3}$$

$$\widehat{\alpha}_2 - 1 = \frac{\sum_{t=1}^n e_t\, y_{t-1} I[y_{t-1} \in D_\tau]}{\sum_{t=1}^n y_{t-1}^2 I[y_{t-1} \in D_\tau]}. \tag{2.4}$$

In order to establish an asymptotic distribution for each of the estimators, we first need to state some auxiliary results. Observe that model (1.1) can be written as

$$y_t - y_{t-1} = (\alpha_1 - 1) y_{t-1} I[y_{t-1} \in C_\tau] + e_t \equiv u_t + e_t, \tag{2.5}$$

where $u_t = (\alpha_1 - 1) y_{t-1} I[y_{t-1} \in C_\tau]$.

Before further discussion, we need to introduce Lemma 2.1 below. As it is a special case of Lemma 3.1 below, we need only to prove Lemma 3.1 in Appendix B.

Lemma 2.1. *Let $\{y_t\}$ be generated by model (1.1). Then $\{y_t\}$ is a β-null recurrent Markov chain with $\beta = \frac{1}{2}$.*

A β-null recurrent Markov chain possesses an invariant measure π_s and there is a variable $T(n)$ keeping track of the number of regenerations at time n. Note that the definitions of $\pi_s(\cdot)$ and $T(n)$ are given in detail in Appendix A below. In this appendix, we have given a motivation for null recurrence in an econometric context and a very brief review of some key facts of the theory. If C_τ