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Testing functional inequalities

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ABSTRACT

This paper develops tests for inequality constraints of nonparametric regression functions. The test statistics involve a one-sided version of L_p-type functionals of kernel estimators ($1 \leq p < \infty$). Drawing on the approach of Poissonization, this paper establishes that the tests are asymptotically distribution free, admitting asymptotic normal approximation. In particular, the tests using the standard normal critical values have asymptotically correct size and are consistent against general fixed alternatives. Furthermore, we establish conditions under which the tests have nontrivial local power against Pitman local alternatives. Some results from Monte Carlo simulations are presented.

1. Introduction

Suppose that we observe $\{(Y_i', X_i')'\}_{i=1}^n$ that are i.i.d. copies from a random vector, $(Y', X')' \in \mathbf{R}^J \times \mathbf{R}^d$. Write $Y_i = (Y_{1i}, \ldots, Y_{Ji})' \in \mathbf{R}^J$ and define $m_j(x) \equiv \mathbf{E}[Y_{ji}|X_i = x],\ j = 1, 2, \ldots, J$. The notation $\equiv$ indicates definition.

This paper focuses on the problem of testing functional inequalities:

$$\begin{aligned} H_0 &: m_j(x) \leq 0 \quad \text{for all } (x, j) \in \mathcal{X} \times \mathcal{J}, \quad \text{vs.} \\ H_1 &: m_j(x) > 0 \quad \text{for some } (x, j) \in \mathcal{X} \times \mathcal{J}, \end{aligned} \tag{1.1}$$

where $\mathcal{X} \subset \mathbf{R}^d$ is the domain of interest and $\mathcal{J} \equiv \{1, \ldots, J\}$. Our testing problem is relevant in various applied settings. For example, in a randomized controlled trial, a researcher observes either an outcome with treatment (W_1) or an outcome without treatment (W_0) along with observable pre-determined characteristics of the subjects (X). Let $D = 1$ if the subject belongs to the treatment group and 0 otherwise. Suppose that assignment to treatment is random and independent of X and that the assignment probability $p \equiv P\{D = 1\},\ 0 < p < 1$, is fixed by the experiment design. Then the average treatment effect $\mathbf{E}(W_1 - W_0|X = x)$, conditional on $X = x$, can be written as

$$\mathbf{E}(W_1 - W_0|X = x) = \mathbf{E}\left[\frac{DW}{p} - \frac{(1-D)W}{1-p} \middle| X = x \right],$$

where $W \equiv DW_1 + (1 - D)W_0$. In this setup, it may be of interest to test whether or not $m(x) \equiv \mathbf{E}(W_1 - W_0|X = x) \leq 0$ for all x.

In economic theory, primitive assumptions of economic models generate certain testable implications in the form of functional inequalities. For example, Chiappori et al. (2006) formulated some testable restrictions in the study of insurance markets. Our tests are applicable for testing their restrictions (e.g. Eq. (4) of Chiappori et al., 2006). Furthermore, our method can be used to test for monotone treatment response (see, e.g. Manski, 2003). For example, testing for a decreasing demand curve for each level of price in treatments and for each value of covariates falls within the framework of this paper.

Our test statistic can also be used to construct confidence regions for a parameter that is partially identified under conditional moment inequalities. See, among many others, Andrews and Shi (forthcoming, 2011), Armstrong (2011), Chernozhukov et al.

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(forthcoming), Chetverikov (2012), and references therein for inference with conditional moment inequalities.

This paper proposes a one-sided L_p approach in testing nonparametric functional inequalities. While measuring the quality of an estimated nonparametric function by its L_p-distance from the true function has long received attention in the literature (see Devroye and Györfi, 1985, for an elegant treatment of the L_1 norm of nonparametric density estimation), the advance of this approach for general nonparametric testing seems to have been rather slow relative to other approaches, perhaps due to its technical complexity.

Csörgő and Horváth (1988) first established a central limit theorem for the L_p-distance of a kernel density estimator from its population counterpart, and Horváth (1991) introduced a Poissonization technique into the analysis of the L_p-distance. Beirlant and Mason (1995) developed a different Poissonization technique and established a central limit theorem for the L_p-distance of kernel density estimators and regressograms from their expected values without assuming smoothness conditions for the nonparametric functions. Giné et al. (2003, GMZ, hereafter) employed this technique to prove the weak convergence of an L_1-distance process indexed by kernel functions in kernel density estimators.

This paper builds on the contributions of Beirlant and Mason (1995) and GMZ to develop methods for testing (1.1). In particular, the tests that we propose are studentized versions of one-sided L_p-type functionals. We show that our proposed test statistic is distributed as standard normal under the least favorable case of the null hypothesis. Thus, our tests using the standard normal critical values have asymptotically correct size. We also show that our tests are consistent against general fixed alternatives and carry out local power analysis with Pitman alternatives. For the latter, we establish conditions under which the tests have nontrivial local power against Pitman local alternatives, including some $n^{-1/2}$-converging Pitman sequences.

Our tests have the following desirable properties. First, our tests do not require the usual smoothness conditions for nonparametric functions for their asymptotic validity and consistency. This is because we do not need pointwise or uniform consistency of an unknown function to implement our tests. For example, a studentized version of our statistic can be estimated without need for controlling the bias. Second, our tests for (1.1) are distribution free under the least favorable case of the null hypothesis where $m_j(x) = 0$, for all $x \in \mathcal{X}$ and for all $j \in \mathcal{J}$, and at the same time have nontrivial power against some, though not all, $n^{-1/2}$-converging Pitman local alternatives. This is somewhat unexpected, given that nonparametric goodness-of-fit tests that involve random vectors of a multi-dimension and have nontrivial power against $n^{-1/2}$-converging Pitman sequences are not often distribution free. Exceptions are tests that use an innovation martingale approach (see, e.g., Khmaladze, 1993, Stute et al., 1998, Bai, 2003, and Khmaladze and Koul, 2004) or some tests of independence (or conditional independence) among random variables (see, e.g., Blum et al., 1961, Delgado and Mora, 2000 and Song, 2009). Third, the local power calculation of our tests for (1.1) reveals an interesting contrast with other nonparametric tests based on kernel smoothers, e.g. Härdle and Mammen (1993) and Horowitz and Spokoiny (2001), where the latter tests are known to have trivial power against $n^{-1/2}$-converging Pitman local alternatives. Our inequality tests can have nontrivial local powers against $n^{-1/2}$-converging Pitman local alternatives, provided that a certain integral associated with local alternatives is strictly positive. On the other hand, it is shown in Section 4 that our equality tests have trivial power against $n^{-1/2}$-converging Pitman local alternatives. Therefore, the one-sided nature of inequality testing is the source of our different local power results. This finding appears new in the literature to the best of our knowledge.

The remainder of the paper is as follows. Section 2 discusses the related literature. Section 3 provides an informal description of our test statistic for a simple case, and establishes conditions under which our tests have asymptotically valid size when the null hypothesis is true and also are consistent against fixed alternatives. We also obtain local power results for the leading cases when $p = 1$ and $p = 2$. In Section 4, we make comparison with functional equality tests and highlight the main differences between testing inequalities and equalities in terms of local power. In Section 5, we report results of some Monte Carlo simulations that show that our tests perform well in finite samples. The proofs of main theorems are contained in Appendix, along with a roadmap for the proof of the main theorem. An online supplement of this paper provides proofs of Lemmas A.1–A.10 in Appendix.

2. Related literature

In this section, we provide details on the related literature. The literature on hypothesis testing involving nonparametric functions has a long history. Many studies have focused on testing parametric or semiparametric specifications of regression functions against nonparametric alternatives. See, e.g., Bickel and Rosenblatt (1973), Härdle and Mammen (1993), Stute (1997), Delgado and González Manteiga (2001), Horowitz and Spokoiny (2001), and Khmaladze and Koul (2004) among many others. The testing problem in this paper is different from the aforementioned papers, as the focus is on whether certain inequality (or equality) restrictions hold, rather than on whether certain parametric specifications are plausible.

When $J = 1$, our testing problem is also different from testing

$H_0 : m(x) = 0 \quad$ for all $x \in \mathcal{X}$, against

$H_1 : m(x) \geq 0$

for all $x \in \mathcal{X}$ with strict inequality for some $x \in \mathcal{X}$.

Related to this type of testing problems, see Hall et al. (1997) and Koul and Schick (1997, 2003) among others. In their setup, the possibility that $m(x) < 0$ for some x is excluded, so that a consistent test can be constructed using a linear functional of $m(x)$. On the other hand, in our setup, negative values of $m(x)$ for some x are allowed under both H_0 and H_1. As a result, a linear functional of $m(x)$ would not be suitable for our purpose.

There also exist some papers that consider the testing problem in (1.1). For example, Hall and Yatchew (2005) and Andrews and Shi (forthcoming, 2011) considered functions of the form $u \mapsto \max\{u, 0\}^p$ to develop tests for (1.1). However, their tests are not distribution free, although they achieve local power against some $n^{-1/2}$-converging sequences. See also Hall and Van Keilegom (2005) for the use of the one-sided L_p-type functionals for testing for monotone increasing hazard rate. None of the aforementioned papers developed test statistics of one-sided L_p-type functionals with kernel estimators like ours. See some remarks of Ghosal et al. (2000, p. 1070) on the difficulty in dealing with one-sided L_p-type functionals with kernel estimators.

In view of Bickel and Rosenblatt (1973), who considered both L_2 and sup tests, a one-sided sup test appears to be a natural alternative to the L_p-type tests studied in this paper. For example, Chernozhukov et al. (forthcoming) considered a sup norm approach in testing inequality constraints of nonparametric functions. Also, it may be of interest to develop sup tests based on a one-sided version of a bootstrap uniform confidence interval of $\hat{g}_n$, similar to Claeskens and van Keilegom (2003). The sup tests typically do not have nontrivial power against any $n^{-1/2}$-converging alternatives, but they may have better power against some "sharp peak" type alternatives (Liero et al., 1998).

Testing for inequality is related to testing for monotonicity since a null hypothesis associated inequality (respectively, monotonicity) can also be framed as that of monotonicity (respectively,