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Rank tests for short memory stationarity[☆]

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ABSTRACT

We propose a rank-test of the null hypothesis of short memory stationarity possibly after linear detrending.

For the level-stationarity hypothesis, the test statistic we propose is a modified version of the popular KPSS statistic, in which ranks substitute the original observations. We prove that the rank KPSS statistic shares the same limiting distribution as the standard KPSS statistic under the null and diverges under I(1) alternatives.

For the trend-stationarity hypothesis, we apply the same rank KPSS statistic to the residuals of a Theil–Sen regression on a linear trend. We derive the asymptotic distribution of the Theil–Sen estimator under short memory errors and prove that the Theil–Sen detrended rank KPSS statistic shares the same weak limit as the least-squares detrended KPSS.

We study the asymptotic relative efficiency of our test compared to the KPSS and prove that it may have unbounded efficiency gains under fat-tailed distributions compensated by very moderate efficiency losses under thin-tailed distributions. For this and other reasons discussed in the body of the article our rank KPSS test turns out to be a valuable alternative to the KPSS for most real-world economic and financial applications.

The weak convergence results and asymptotic representations proved in this article should interest a wider audience than that concerned with stationarity testing, as they extend to ranks analogous invariance principles widely used in unit-root econometrics.

1. Introduction

We propose a rank test for the null hypothesis of short memory[1] stationarity (often referred to as *level-stationarity*), and a rank test for the null of short memory stationarity on a linear trend (also *trend-stationarity*).

Our test for level-stationarity is based on the well-known KPSS statistic (Kwiatkowski et al., 1992) applied to the ranks of the observations. The asymptotic distribution of our rank KPSS (RKPSS) statistic under the null is the same as that of the original KPSS and, thus, any software package implementing the KPSS test implements our RKPSS test as well. The advantages of our RKPSS test over Kwiatkowski et al.'s (1992) are: (i) the existence of moments is not necessary, substituted by absolute continuity, (ii) the test statistic is invariant to monotonic transformations of the data, and (iii) the asymptotic relative efficiency of the RKPSS with respect to the KPSS test is moderately smaller than one under thin-tailed distribution, but may increase without bound under fat-tailed distributions. Property (ii) is very valuable, since economic data are often transformed through logarithms or other strictly monotonic maps (i.e., Box–Cox, inverse logit, etc.). Since weak dependence (in its connotation of strong mixing) and stationarity are not altered by such transformations, it is highly desirable for the outcome of the test not to depend on their choice.

The idea for the RKPSS test was inspired by a recent work by de Jong et al. (2007), who apply the KPSS statistic to the signs of the median-adjusted observations. Unfortunately, their approach leads to a test with competitive power properties only under extremely fat-tailed distributions as the simulations in Section 5 show. Furthermore, de Jong et al. (2007) do not provide a study of the (asymptotic) efficiency of their test and do not derive the theory for a test of trend-stationarity.

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[1] Throughout the paper the term 'short memory' will be used as synonymous to 'weakly dependent' or 'short-range dependent' and implemented through a strong mixing condition.

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Our test for trend-stationarity is based on the RKPSS statistic applied to the residuals of a regression of the observations on a linear trend. The regression slope is estimated using the Theil–Sen estimator proposed by Theil (1950) and generalized by Sen (1968). The Theil–Sen estimator is simple to compute, robust to fat-tails and guarantees that, under the null, the asymptotic distribution of the detrended RKPSS statistic is the same as that of the detrended KPSS statistic.

The weak convergence results and asymptotic representations we prove in this paper should interest a wider audience than stationarity test users. Indeed, they represent generalizations to ranks of many results commonly used in unit-root econometrics such as those listed in the widely cited paper by Phillips and Perron (1988). In particular, the asymptotic distribution of the Theil–Sen estimator under strong mixing, derived in Section 4, may be of interest to all those who seek a robust and easy way to detrend fat-tailed observations.

The plan of the paper is as follows. In Section 2 we provide the asymptotic theory for the RKPSS statistic both under the null and under the alternative of integration. In Section 3 we study the asymptotic relative efficiency of our test compared to the KPSS. Section 4 derives the asymptotic distribution of the test for trend-stationarity under the null and under the alternative of integration. Section 5 presents a battery of simulations for comparing the finite-sample properties of the KPSS and RKPSS statistics, also with the IKPSS statistic of de Jong et al. (2007). Finally, in Section 7 we give some concluding remarks and directions for future research. All proofs can be found in the Appendix.

2. Rank KPSS for level stationarity

Let the observed time series be a sample path of the real random sequence $\{X_1, X_2, \ldots, X_T\}$ and let

$$R_{T,t} = \sum_{i=1}^{T} \mathbb{I}_{\{X_i \le X_t\}}, \quad \text{for } t = 1, \ldots, T, \tag{1}$$

with $\mathbb{I}_A$ the indicator function of the set A, be the *rank* of X_t among $\{X_1, \ldots, X_T\}$. Notice that the arithmetic mean of the rank sequence $\{R_{T,1}, \ldots, R_{T,T}\}$ is $(T+1)/2$ and does not depend on the data.

The test statistic we propose is the KPSS applied to the ranks of the observations. So, let $S_{T,t}$ be the sequence of demeaned partial sums:

$$S_{T,t} = \sum_{i=1}^{t} \left(\frac{R_{T,i}}{T} - \frac{T+1}{2T} \right). \tag{2}$$

Notice that the KPSS statistic is invariant to scale transformations, so working with $R_{T,i}/T$ rather than $R_{T,i}$ turns out to generate the same statistic.

In analogy with Kwiatkowski et al. (1992), define the random quantity

$$\eta_T^R = T^{-2} \sum_{t=1}^{T} S_{T,t}^2 \tag{3}$$

and the RKPSS test statistic for level-stationarity as $\hat{\eta}_T^R = \eta_T^R / \hat{\sigma}_T^2$, where $\hat{\sigma}_T^2$ is a HAC estimator of the long-run variance of the process $\{R_{T,t}/T\}$:

$$\hat{\sigma}_T^2 = \frac{1}{T} \sum_{s=1}^{T} \sum_{t=1}^{T} k\left(\frac{s-t}{\gamma_T}\right) \left[\frac{R_{T,s}}{T} - \frac{T+1}{2T}\right] \left[\frac{R_{T,t}}{T} - \frac{T+1}{2T}\right], \tag{4}$$

with $k(\cdot)$ a symmetric kernel function and γ_T a bandwidth parameter.

We state here two assumptions that will be useful in the rest of the paper.

Assumption 1. Short memory stationarity

1. $\{X_t\}$ is a strictly stationary random sequence.
2. $\{X_t\}$ is strong mixing with parameter $\alpha(m) = O(m^{-\upsilon})$, $\upsilon > 2$.
3. The marginal distribution of X_t, $F(\cdot)$, is non-degenerate absolutely continuous with density $f(\cdot)$.

Assumption 2. Regularity of the kernel function $k(\cdot)$

1. $\int_{-\infty}^{\infty} |\psi(z)|\,\mathrm{d}z < \infty$, where $\psi(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} k(x)\exp(-\mathrm{i}zx)\,\mathrm{d}x$.
2. $k(\cdot)$ is continuous at all but a finite number of points, $k(x) = k(-x)$, $|k(x)| < l(x)$ where $l(x)$ is non-increasing with $\int_0^\infty |l(x)|\,\mathrm{d}x < \infty$, and $k(0) = 1$.
3. $\gamma_T/\sqrt{T} \to 0$ and $\gamma_T \to \infty$ as $T \to \infty$.

Define the population counterpart of Spearman's rank autocorrelation coefficient as $\rho_{i,j} := 12\,\mathbb{E}\left\{\left[F(X_i) - 1/2\right]\left[F(X_j) - 1/2\right]\right\} = \mathbb{C}\mathrm{orr}\left(F(X_i), F(X_j)\right)$.

The following theorem gives the asymptotic distribution of the test statistic under the null of strong mixing stationarity.[2]

Theorem 1. *Under Assumption 1,*

$$\eta_T^R \Rightarrow \sigma^2 \int_0^1 V(r)^2\,\mathrm{d}r, \tag{5}$$

with V standard Brownian bridge and

$$\sigma^2 = \frac{1}{12}\left[1 + 2\sum_{k=2}^{\infty} \rho_{1,k}\right]; \tag{6}$$

furthermore, uniformly in $r \in [0, 1]$,

$$T^{-1/2} S_{T,\lfloor rT \rfloor} = T^{-1/2}\left\{\sum_{i=1}^{\lfloor rT \rfloor} F(X_i) - \frac{\lfloor rT \rfloor}{T}\sum_{i=1}^{T} F(X_i)\right\} + \mathrm{O}_p(T^{-1/2}). \tag{7}$$

Under Assumptions 1 and 2,

$$\hat{\eta}_T^R \Rightarrow \int_0^1 V(r)^2\,\mathrm{d}r. \tag{8}$$

Under the alternative that X_t is an I(1) process, we have the following result, that proves the consistency of our test under the most econometrically relevant alternative.

Theorem 2. *Suppose there exists a strictly monotone (Borel) function* $g : \mathbb{R} \mapsto \mathbb{R}$ *such that* $T^{-1/2} g(X_{\lfloor rT \rfloor}) \Rightarrow \omega W(r)$, $r \in [0, 1]$, *where* ω *is a strictly positive real number and W a standard Brownian motion on* $[0, 1]$, *then*

$$\frac{\eta_T^R}{T} \Rightarrow \int_0^1 \left[\int_0^s R_0(r)\,\mathrm{d}r\right]^2 \mathrm{d}s, \tag{9}$$

with $R_0(r) = \int_0^1 \mathbb{I}_{\{W(u) \le W(r)\}}\,\mathrm{d}u - \frac{1}{2}$, *and*

$$\hat{\sigma}_T^2 \le \frac{1}{T}\sum_{s=1}^{T}\sum_{t=1}^{T} k\left(\frac{s-t}{\gamma_T}\right) = O(\gamma_T). \tag{10}$$

[2] Weak convergence will be denoted by $\Rightarrow$ and is to be intended with respect to the Skorohod J_1-topology on the space $D[0, 1]$ of *càdlàg* functions on the unit interval. $\lfloor x \rfloor$ indicates the integer part (or *floor*) of x.