Computers and Structures 161 (2015) 1–16

Contents lists available at ScienceDirect

Computers and Structures

journal homepage: www.elsevier.com/locate/compstruc

Formulation of the boundary element method in the wavenumber–frequency domain based on the thin layer method

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ARTICLE INFO

Article history:
Received 9 May 2015
Accepted 13 August 2015
Available online 8 September 2015

Keywords:
Boundary element method
Thin-layer method
Soil-structure
Invariant structures

ABSTRACT

This paper links the boundary element method (BEM) and the thin-layer method (TLM) in the context of structures that are invariant in one direction and for which the equations of motion can be formulated in the wavenumber–frequency domain (2.5D domain). The proposed combination differs from previous formulations in that one of the inverse Fourier transforms and the Green's functions (GF) integrals are obtained in closed form. This strategy is not only supremely efficient, but also avoids singularities when the collocation point belongs to the integrating boundary element, and provides accurate evaluations of the coefficients of the boundary element matrices.

1. Introduction

It is often the case that a domain can be idealized as a longitudinally invariant medium, i.e., a structure whose cross section remains constant along a given direction, say in direction y. For instance, in the case of vibrations induced by moving vehicles, it is often convenient to idealize the road, track or tunnel as a structure whose geometry is invariant in the longitudinal direction [1]. In that case, after carrying out a Fourier transform from the Cartesian spatial coordinate y to the horizontal wavenumber k_y, the analysis of the three dimensional structure can be reduced to a series of 2D problems. This type of analysis is referred to as a two-and-a-half dimensional (2.5D) problem and is normally cast in the wavenumber–frequency domain (k_y, ω).

Furthermore, whenever the domain under consideration is unbounded (e.g., soil-structure interaction problems), the radiation of waves at infinity must be accounted for. The boundary element method (BEM) intrinsically accounts for the radiation condition and therefore is one of the tools most commonly used in these situations. The BEM requires the availability of the so called fundamental solution (or Green's functions – GF), which in the vast majority of cases are those for a homogeneous, complete space (i.e. the Stokes–Kelvin problem), and rarely those of layered spaces. The reason for this is that in the 2.5D domain, the GF for homogeneous whole-spaces are known in analytical form [2], while the GF for layered spaces can only be obtained via numerical methods such as transfer matrices [3,4], stiffness matrices [5], or the thin-layer method (TLM) [6].

Formulations for the 2.5D BEM were previously given in Refs. [7,8] using the whole-space GF and in Ref. [9] using the GF for layered spaces obtained via the stiffness matrix method. In this work, we present a very efficient alternative formulation based on the Green's functions obtained with the TLM (2.5D BEM + TLM). When compared with the formulations in [7,8], the proposed procedure has the enormous advantage of avoiding the discretization of the free-surface of a half-space and of the interfaces between material layers, because layering is considered automatically in the definition of the GF. It accomplishes this at the expense of more elaborate computations to obtain the GF. When compared with the work presented in [9], the 2.5D BEM + TLM approach described herein replaces the discrete numerical Fourier inversion in k_y by exact modal summations, which requires solving a narrowly-banded quadratic eigenvalue problem. This circumvents the need for an appropriate wavenumber step for k_x and thus avoids the problems of spatial periodicity, wrap-around and aliasing. Another advantage of the 2.5D BEM + TLM is the avoidance of the numerical integration of the Green's functions over the boundary elements, which is replaced by modal summations, a feature that circumvents the complication entailed by the singularities contained in the GF.

This article is organized as follows: in Section 2 the TLM is reviewed and the expressions for the calculation of the 2.5D displacements and stresses are obtained; in Section 3 the direct calculation of the coefficients of the boundary element matrices

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http://dx.doi.org/10.1016/j.compstruc.2015.08.012
0045-7949/

is addressed; finally, in Section 4 the proposed procedure is validated by means of some examples.

2. TLM in the 2.5D domain – Green's functions

The TLM is an efficient semi-analytical method for the calculation of the fundamental solutions (i.e., GF) of layered media. It consists in expressing the displacement field in terms of a finite element expansion in the direction of layering together with analytical descriptions for the remaining directions. Though initially it was limited to domains of finite depth, paraxial boundaries were developed and coupled to the TLM in order to circumvent this limitation [10]. More recently, perfectly matched layers (PML) have been proposed and shown to be more accurate than paraxial boundaries for the simulation of unbounded domains [11].

The TLM has been formulated in the space-frequency domain (2D, 3D) [12] and in the wavenumber-time domain [13]. It has also been formulated in the 2.5D domain (x, k_y, ω) [14], but solely in terms of displacements elicited by applied forces, and has been coupled to the BEM in the context of 3D axisymmetric structures [15]. This section presents the derivation of the expressions for the displacements, their spatial derivatives and the stresses anywhere, both in the wavenumber domain (k_x, k_y, ω) and in the 2.5D domain (x, k_y, ω). Variables with an over-bar or tilde represent field quantities in the (k_x, k_y, ω) domain, while variables denoted without diacritical marks represent fields in the mixed (x, k_y, ω) domain.

2.1. Displacements in the wavenumber domain (k_x, k_y, ω)

In [14], a TLM formulation is presented in which the displacements elicited by various kinds of loads acting within layered media are obtained in the (k_x, k_y, ω) and (x, k_y, ω) domains. Following that work, after discretizing a layered domain into thin-layers and applying the principle of weighted residuals, we obtain a matrix equation for each thin-layer of the form

$$\overline{\mathbf{P}} = \left[k_x^2\mathbf{A}_{xx} + k_x k_y \mathbf{A}_{xy} + k_y^2 \mathbf{A}_{yy} + i\left(k_x \mathbf{B}_x + k_y \mathbf{B}_y\right) + \left(\mathbf{G} - \omega^2\mathbf{M}\right)\right]\overline{\mathbf{U}} \quad (1)$$

where the vectors $\overline{\mathbf{P}}$ and $\overline{\mathbf{U}}$ contain, respectively, the external tractions $\overline{p}_\alpha(k_x, k_y, \omega)$ and displacements $\overline{u}_\alpha(k_x, k_y, \omega)$ at the nodal interfaces, and where the remaining boldface variables are matrices that depend solely on the material properties of the thin-layers. These matrices are listed in Appendix A for the case of cross-anisotropic materials. The variables k_x and k_y represent the horizontal wavenumbers in the transverse and longitudinal directions, respectively, ω represents the angular frequency, the index $\alpha (= x, y, z)$ represents the direction of the nodal displacement and/or traction, and $i = \sqrt{-1}$ is the imaginary unit.

By means of a similarity transformation, Eq. (1) can be changed into

$$\tilde{\mathbf{p}} = \left[k_x^2\mathbf{A}_{xx} + k_x k_y \mathbf{A}_{xy} + k_y^2 \mathbf{A}_{yy} + k_x \tilde{\mathbf{B}}_x + k_y \tilde{\mathbf{B}}_y + \left(\mathbf{G} - \omega^2\mathbf{M}\right)\right]\tilde{\mathbf{u}} \quad (2)$$

where $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{u}}$ are obtained from $\overline{\mathbf{P}}$ and $\overline{\mathbf{U}}$ by multiplying every third row by $-i$ and where $\tilde{\mathbf{B}}_x$ and $\tilde{\mathbf{B}}_y$ are obtained from $\mathbf{B}_x$ and $\mathbf{B}_y$ by simply reversing the sign of every third column. Eq. (2) is advantageous over Eq. (1) because the matrices therein are symmetric while in Eq. (1) they are not.

After assembling the thin-layer matrices for all the thin-layers, we obtain the global system of equations with the same configuration as Eq. (2), and although it can easily be solved for $\tilde{\mathbf{u}}$, we choose to follow an alternative and more convenient approach. In fact, the direct numerical solution of Eq. (2) for $\tilde{\mathbf{u}}$ (or $\tilde{\mathbf{p}}$) precludes the analytical evaluation of the inverse Fourier transforms from the (k_x, k_y, ω) domain to the (x, k_y, ω) domain, and consequently renders the TLM an inefficient method when compared with the stiffness matrix approach. For this reason, an alternative approach is followed wherein we find a modal basis with which we can calculate $\tilde{\mathbf{u}}$ and/or $\tilde{\mathbf{p}}$ through modal superposition. This procedure enables the analytical transformation of $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{p}}$ to the desired 2.5D domain, which constitutes an enormous advantage.

Without entering into lengthy details, the modal basis is found by solving a quadratic eigenvalue problem in k of the form [14]

$$\left[k^2\mathbf{A}_{xx} + k\tilde{\mathbf{B}}_x + \left(\mathbf{G} - \omega^2\mathbf{M}\right)\right]\boldsymbol{\phi} = \mathbf{0} \quad (3)$$

Rearranging the matrices in this eigenvalue problem by degrees of freedom (first x, then y and finally z), we observe that these matrices attain the following structures

$$\mathbf{A}_{xx} = \begin{bmatrix} \mathbf{A}_x & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{A}_y & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{A}_z \end{bmatrix} \qquad \tilde{\mathbf{B}}_x = \begin{bmatrix} \mathbf{O} & \mathbf{O} & \mathbf{B}_{xz} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{B}_{xz}^T & \mathbf{O} & \mathbf{O} \end{bmatrix}$$
$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_x & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{G}_y & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{G}_z \end{bmatrix} \qquad \mathbf{M} = \begin{bmatrix} \mathbf{M}_x & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{M}_y & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{M}_z \end{bmatrix} \quad (4)$$

Because of the special structure of these matrices, the eigenvalue problem in Eq. (3) can be decoupled into the following two eigenvalue problems

$$\left\{k^2\begin{bmatrix} \mathbf{A}_x & \mathbf{O} \\ \mathbf{O} & \mathbf{A}_z \end{bmatrix} + k\begin{bmatrix} \mathbf{O} & \mathbf{B}_{xz} \\ \mathbf{B}_{xz}^T & \mathbf{O} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_x & \mathbf{O} \\ \mathbf{O} & \mathbf{C}_z \end{bmatrix}\right\}\begin{bmatrix} \boldsymbol{\phi}_x \\ \boldsymbol{\phi}_z \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$
$$\left(k^2\mathbf{A}_y + \mathbf{C}_y\right)\boldsymbol{\phi}_y = \mathbf{0} \quad (5)$$

which correspond to the generalized Rayleigh and generalized Love eigenvalue problems. The first eigenvalue problem has $2N_R$ solutions while the second has $2N_L$ solutions, with N_R and N_L being the dimension of the corresponding matrices. For the calculation of the responses, only the solutions that correspond to eigenvalues with negative imaginary components are considered, because only these entail waves that carry energy away from the source. Hence only N_R solutions of the Rayleigh problem and only N_L solutions of the Love problem are considered. Based on the eigensolutions, the displacements $\overline{u}_{\alpha\beta}^{(mn)}$ at the mth nodal interface in direction α due to a unit load applied at the nth nodal interface in direction β are calculated by modal superposition as listed in Table 1, with the coefficients K_{ij} given in Table 2.

The displacements at an interior horizontal plane of the ith thin-layer are obtained by vertical interpolation of the nodal values, i.e.,

$$\overline{u}_{\alpha\beta}(z) = \sum_{j=1}^{nn} N_j(z)\,\overline{u}_{\alpha\beta(j)}^{(i)} \quad (6)$$

with nn being the number of nodal interfaces within each thin-layer ($nn = 2$ for linear expansion, $nn = 3$ for quadratic expansion, etc.), $\overline{u}_{\alpha\beta(j)}^{(i)}$ the nodal displacement of the jth nodal interface of the considered thin-layer, and $N_j(z)$ the corresponding shape function.

Table 1
Nodal displacements in frequency–wavenumber domain.

$\overline{u}_{xx}^{(mn)} = \sum_j^{N_R} K_{3j}\,\phi_{xj}^{(m)}\phi_{xj}^{(n)} + \sum_j^{N_L} K_{4j}\phi_{yj}^{(m)}\phi_{yj}^{(n)}$	
$\overline{u}_{yy}^{(mn)} = \sum_j^{N_R} K_{4j}\,\phi_{xj}^{(m)}\phi_{xj}^{(n)} + \sum_j^{N_L} K_{3j}\,\phi_{yj}^{(m)}\phi_{yj}^{(n)}$	
$\overline{u}_{xy}^{(mn)} = \sum_j^{N_R} K_{2j}\,\phi_{xj}^{(m)}\phi_{xj}^{(n)} - \sum_j^{N_L} K_{2j}\,\phi_{yj}^{(m)}\phi_{yj}^{(n)} = \overline{u}_{yx}^{(mn)}$	
$\overline{u}_{xz}^{(mn)} = -i\sum_j^{N_R} K_{5j}\,\phi_{xj}^{(m)}\phi_{zj}^{(n)}$	$\overline{u}_{zx}^{(mn)} = i\sum_j^{N_R} K_{5j}\,\phi_{zj}^{(m)}\phi_{xj}^{(n)} = -\overline{u}_{xz}^{(nm)}$
$\overline{u}_{yz}^{(mn)} = -i\sum_j^{N_R} K_{6j}\,\phi_{xj}^{(m)}\phi_{zj}^{(n)}$	$\overline{u}_{zy}^{(mn)} = i\sum_{j=1}^{N_R} K_{6j}\,\phi_{zj}^{(m)}\phi_{xj}^{(n)} = -\overline{u}_{yz}^{(nm)}$
$\overline{u}_{zz}^{(mn)} = \sum_j^{N_R} K_{1j}\,\phi_{zj}^{(m)}\phi_{zj}^{(n)}$	