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Distribution-free tests of stochastic monotonicity☆

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ABSTRACT

This article proposes a nonparametric test of monotonicity for conditional distributions and its moments. Unlike previous proposals, our method does not require smooth estimation of the derivatives of nonparametric curves. Distinguishing features of our approach are that critical values are pivotal under the null in finite samples and that the test is invariant to any monotonic continuous transformation of the explanatory variable. The test statistic is the sup-norm of the difference between the empirical copula function and its least concave majorant with respect to the explanatory variable coordinate. The resulting test is able to detect local alternatives converging to the null at the parametric rate $n^{-1/2}$, with n the sample size. The finite sample performance of the test is examined by means of a Monte Carlo experiment and an application to testing intergenerational income mobility.

1. Introduction

Let (Y, X) be a bivariate random vector taking values in $\mathcal{Y}\times\mathcal{X}\subseteq\mathbb{R}^2$ and with induced joint distribution

$$F(y,x)=\int_{-\infty}^{x}F_{Y|X}(y|\bar{x})F_X(d\bar{x}),\quad (y,x)\in\mathcal{Y}\times\mathcal{X}, \tag{1}$$

where $F_{Y|X}$ is the conditional distribution function of Y given X and, henceforth, F_ξ denotes the marginal cumulative distribution function (cdf) of the generic random variable (r.v.) ξ. This article proposes a nonparametric test for the monotonicity of $F_{Y|X}$ with respect to the covariate X. That is, the null hypothesis is

$$H_0 : F_{Y|X}(y|\cdot)\in\mathcal{M}\quad\text{for each } y\in\mathcal{Y}, \tag{2}$$

where $\mathcal{M}$ is the set of monotonically non-increasing functions with support $\mathcal{X}$, *i.e.,*

$$\mathcal{M}=\left\{m:\mathcal{X}\subseteq\mathbb{R}\to\mathbb{R}\text{ s.t. } m\left(x'\right)\geq m\left(x''\right)\text{ for } x'\leq x''\right\}.$$

We consider omnibus tests where the alternative hypothesis, H_1, is the negation of H_0. The discussion and results below obviously apply to the monotonically non-decreasing case *mutatis mutandis*. This testing problem has been recently addressed by Lee et al. (2009), LLW henceforth, which generalizes the test of monotonicity for regression functions proposed by Ghosal et al. (2000), GSV henceforth.

Testing monotonicity is interesting, first of all, because estimators of nonparametric monotonic curves can be obtained without imposing smoothness restrictions. See e.g. Brunk (1958) and the monograph by Barlow et al. (1972). The efficiency of these isotonic estimators can be improved when it is additionally known that the nonparametric curve is smooth. See e.g. Mammen (1991) and Mukerjee (1988).

The null hypothesis H_0 states a stochastic dominance assumption on subpopulations defined by means of the values taken by the covariate X. For instance, when $Y = Y(t+1)$ and $X = Y(t)$, for a Markov process $\{Y(t)\}_{t\in\mathbb{Z}}$, this generalizes the usual stochastic dominance concept for the transition probabilities of Markov chains to a continuous state space, see e.g. Kadi et al. (2009) for a discussion. Stochastic monotonicity plays a crucial role in stochastic dynamic programming in order to ensure the uniqueness of the equilibrium solution. See Chapters 9 and 12 of Lucas and Stokey (1989). This property is often assumed when modeling industrial economics dynamics. See e.g. Ericson and Pakes (1995), Pakes (1986) or Olley and Pakes (1996). Monotonicity is also an

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important identification assumption in many nonparametric and semiparametric settings, see Matzkin (1994) for a survey and Aguirregabiria (2010), Banerjee et al. (2009), Lewbel and Linton (2007), and Tanaka (2008) for some recent applications. The monotonicity of the intergenerational transition function is also worth testing in the analysis of intergenerational mobility; i.e. having a parent from a high social-economic status is never worse than having one with a lower status. This testing problem has been considered by LLW and Dardanoni et al. (2012) using different data sets. Many theories in finance also imply monotonic patterns in expected returns and other financial variables, see e.g. Boudoukh et al. (1999) and Richardson et al. (1992). Recently, Patton and Timmermann (2009) have proposed tests of monotonicity and have applied them to test whether expected returns are monotonically decreasing or monotonically increasing in securities' risk or liquidity characteristics. The tests presented in this article can be used to extend their results to a continuous covariate.

The null hypothesis H_0 implies that for any non-increasing function in the second argument $\gamma : \mathcal{Y} \times \mathcal{X} \to [0, \infty)$,

$$H_0^{\gamma} : \mathbb{E}\left(\gamma\left(Y, X\right) | X = \cdot\right) = \int_{\mathcal{Y}} \gamma\left(y, \cdot\right) F_{Y|X}\left(dy | \cdot\right) \in \mathcal{M}. \qquad (3)$$

The function γ could be non-parametric. In fact, H_0^{γ} with nonparametric γ is crucial in modeling under asymmetric information. For instance, in signaling models, the analysis is conducted by a monotonicity property; e.g. more talented workers buy higher education (Spence, 1973) or work faster (Akerlof, 1976) than their less talented competitors. Monotonicity also plays a crucial role in adverse selection; e.g. Akerlof (1970) "lemons" model, where higher prices in the used car market results in a higher average quality of the cars available. Additional examples of the role of monotonicity can be found in the literature on search, advertising and bidding. See Milgrom (1981) for discussion.

Testing H_0^{γ} with γ known or parametric is interesting on its own in many circumstances. Testing the monotonicity of regression curves is a natural hypothesis to test. In fact, the monotonicity of reduced form mean responses forms a basis for the identification of non-parametric structural relations. See Manski and Pepper (2000). Monotonicity of a regression function is also essential for the root-n consistent estimation of convolution density estimators in Escanciano and Jacho-Chávez (2012) and references therein. The test for H_0^{γ} with γ known of GSV, extended to testing H_0 by LLW, as well as the vast majority of existing monotonicity tests, rely on the assumption that the nonparametric curve is smooth enough, and the tests are based on some kind of smooth nonparametric estimator of the first derivatives. See also previous proposals by Bowman et al. (1998), Schlee (1982), or Hall and Heckman (2000). The performance of these tests depends on the satisfaction of several assumptions on the nonparametric curve whose monotonicity is tested, as well as other underlying nonparametric curves, despite the nuisance of suitably choosing some smoothing parameter.

In this article, rather than looking at the first derivative of the curve, we pay attention to its integral. To that end, we introduce the copula function

$$C\left(u, v\right) := F\left(F_Y^{-1}\left(u\right), F_X^{-1}\left(v\right)\right), \quad \left(u, v\right) \in [0, 1]^2,$$

where F_{ξ}^{-1} denotes the quantile function, i.e. the generalized inverse $F_{\xi}^{-1}(u) := \inf\{t \in \mathbb{R} : F_{\xi}(t) \geq u\}$, $u \in [0, 1]$, associated to the cdf F_{ξ}. We shall assume that F_X and F_Y are continuous. Hence, from (1) we can write

$$C\left(u, v\right) = \int_0^v F_{Y|X}\left(F_Y^{-1}\left(u\right) \middle| F_X^{-1}(\bar{v})\right) d\bar{v}, \quad \left(u, v\right) \in [0, 1]^2.$$

Let $\mathcal{C}$ be the set of concave functions on [0, 1]. The condition

$$C\left(u, \cdot\right) \in \mathcal{C} \quad \text{for each } u \in [0, 1] \qquad (4)$$

is necessary and sufficient for H_0. Sufficiency is guaranteed because a concave function has non-increasing derivatives, and necessity is proved, for instance, in Apostol (1967, Theorem 2.9).

Therefore, the null hypothesis can be alternatively characterized using the least concave majorant (l.c.m) operator, $\mathcal{T}$ say, applied to the explanatory variable coordinate. That is, the l.c.m of $C\left(u, \cdot\right)$ for each $u \in [0, 1]$ fixed, $\mathcal{T}C\left(u, \cdot\right)$, is the function satisfying the following two properties: (i) $\mathcal{T}C\left(u, \cdot\right) \in \mathcal{C}$ and (ii) if there exists $h \in \mathcal{C}$ with $h \geq C\left(u, \cdot\right)$, then $h \geq \mathcal{T}C\left(u, \cdot\right)$. Henceforth, $\mathcal{T}C$ denotes the function obtained by applying the operator $\mathcal{T}$ to the function $C\left(u, \cdot\right)$ for each $u \in [0, 1]$. Thus, we can alternatively write H_0 as

$$H_0 : \mathcal{T}C = C. \qquad (5)$$

Obviously, the greatest convex minorant must be used for characterizing H_0 in the monotonically non-decreasing case. The copula function C, and therefore $\mathcal{T}C$, can be estimated by its sample analog. Notice that the slope of $\mathcal{T}C$ with respect to the second coordinate is a restricted version of $F_{Y|X}$, i.e. concave with respect to X. Our approach is then related to the classical literature on inference under shape restrictions. Grenander (1956) first found that the slope of the l.c.m of the empirical distribution is the maximum likelihood estimator of a monotonic non-increasing probability density. Chernoff (1964) applied Grenander's ideas to the estimation of a mode and Prakasa Rao (1969) to the estimation of an unimodal probability density. Brunk (1958) extended this idea to estimating a monotonic (isotonic) regression function, see Barlow et al. (1972) for a monograph on isotonic regression. These ideas are behind the classical DIP test of unimodality proposed by Hartigan and Hartigan (1985). More recently, Durot (2003) has used the difference between the empirical integrated regression function and its l.c.m. for testing monotonicity of a regression curve in a fixed regressor setting with independent and identically distributed (iid) errors. The fixed regressor assumption is rather restrictive and rules out most applications of interest in economics. Moreover, a näive extension of Durot's (2003) method to stochastic regressors is not valid because the integrated regression function is not necessarily concave or convex when the regression function is monotone.

Estimates of the l.c.m. of the copula process are used in this article for testing monotonicity of the conditional cdf, only assuming continuity of the marginal distributions. Distinguishing features of our approach are that test's critical values are pivotal under the null and that the test is invariant to any monotonic continuous transformation of the explanatory variable in finite samples. The former feature is inherited from the use of the copula process, the latter should be a minimal requirement for any test of monotonicity. Our proposal permits us to relax different smoothness assumptions on the underlying nonparametric curves imposed by LLW and related tests. In particular, with our approach there is no need to estimate derivatives of nonparametric conditional curves, which requires a bandwidth choice.

The rest of the article is organized as follows. The test is discussed in Section 2. Section 3 presents an asymptotic test for H_0^{γ} with γ known. The results of a Monte Carlo study are reported in Section 4, together with an application of the new test to studying intergenerational income mobility. Mathematical proofs are gathered in a technical mathematical Appendix at the end of the article.

2. Testing stochastic monotonicity

Given independent copies $\mathcal{Z}_n := \{(Y_i, X_i)\}_{i=1}^n$ of (Y, X), the natural estimator of $C\left(u, v\right)$ is the empirical copula process

$$C_n\left(u, v\right) := \frac{1}{n} \sum_{i=1}^n 1_{\{F_{Yn}(Y_i) \leq u\}} 1_{\{F_{Xn}(X_i) \leq v\}}, \quad \left(u, v\right) \in [0, 1]^2, \qquad (6)$$