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Estimation of stable distributions by indirect inference

René Garcia [a,*], Eric Renault [b,c], David Veredas [d]

[a] EDHEC Business School, France
[b] University of North Carolina at Chapel Hill, USA
[c] CIRANO and CIREQ, Canada
[d] ECARES, Solvay Brussels School of Economics and Management, Université libre de Bruxelles, Belgium

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ABSTRACT

This article deals with the estimation of the parameters of an α-stable distribution with indirect inference, using the skewed-t distribution as an auxiliary model. The latter distribution appears as a good candidate since it has the same number of parameters as the α-stable distribution, with each parameter playing a similar role. To improve the properties of the estimator in finite sample, we use constrained indirect inference. In a Monte Carlo study we show that this method delivers estimators with good properties in finite sample. We provide an empirical application to the distribution of jumps in the S&P 500 index returns.

1. Introduction

The α-stable distribution has been widely used for fitting data in which extreme values are frequent. As shown in early work by Mandelbrot (1963) and Fama (1965a), it accommodates heavy-tailed financial series, and therefore produces more reliable measures of tail risk. The α-stable distribution is also able to capture skewness in a distribution, which is another characteristic feature of financial series. The distribution is also preserved under convolution. This property is appealing when considering portfolios of assets, especially when the skewness and fat tails of returns are taken into account to determine the optimal portfolio.[1] Stable processes have recently been used in the high frequency microstructure literature by Ait-Sahalia and Jacod (2007, 2008) who proposed volatility estimators for some processes built from the sum of a stable process and another Levy process.

To estimate the parameters of an α-stable distribution we propose to use indirect inference (see Smith, 1993; Gouriéroux et al., 1993, GMR hereafter), a method particularly suited to situations where the model of interest is difficult to estimate but relatively easy to simulate. Indeed, the α-stable density function does not have a closed-form expression and is only characterized as an integral difficult to compute numerically, making ML estimation not very appealing in practice.[2] However, several methods are available to simulate α-stable random variables, such as the one described in Chambers et al. (1976).

Indirect inference involves the use of an auxiliary model. Auxiliary parameters are recovered through maximization of the pseudo-likelihood of a model based on the fictitious *i.i.d.* sampling in a skewed-t distribution of Fernández and Steel (1998).[3] It is a Student-t with an inverse scale factor in the positive and negative orthants, allowing for asymmetries. The distribution has four parameters which have a one-to-one correspondence with the parameters of the α-stable distribution. There is a clear and interpretable matching between the two sets, parameter by parameter.[4]

* Corresponding address: EDHEC-393, Promenade des Anglais BP3116 06202 Nice Cedex 3, France. Tel.: +33 0 493189966; fax: +33 0 493830810.
E-mail addresses: rene.garcia@edhec.edu (R. Garcia), renault@email.unc.edu (E. Renault), dveredas@ulb.ac.be (D. Veredas).

1 Basic references on the α-stable distribution are Feller (1971), Zolotarev (1986) and Samorodnitsky and Taqqu (1994). Its properties motivate its use in the modelling of financial series in particular by Carr et al. (2002) and Mittnik et al. (2000). For value-at-risk applications, see in particular Bassi et al. (1998) and Mittnik et al. (1998). For portfolio allocation with stable distributions, see Fama (1965b), Bawa et al. (1979), and Ortobelli et al. (2002).

2 Nevertheless, DuMouchel (1973) has shown that the maximum likelihood (ML hereafter) estimator is consistent, asymptotically normal and reaches the Cramer–Rao efficiency bound.

3 Hansen (1994) also proposes a skewed version of the Student-t. The way skewness is introduced differs from that of Fernández and Steel (1998).

4 During the course of this project, we were made aware by Lombardi that Lombardi and Calzolari (2008) use the same auxiliary model to estimate a stable distribution. The two projects were conducted independently and differ in several respects.

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Our application of indirect inference is innovative in two respects. First, following McCulloch (1986) in the context of matching quantiles, we actually perform a constrained version of indirect inference, introducing an a priori constraint on one auxiliary parameter to match, namely the number of degrees of freedom of the Student-t. The theory for such constrained indirect inference (CII hereafter) has been developed in a general context by Calzolari et al. (2004) (CFS hereafter). Second, we stress in our application that the α-stable simulator need not to take into account the actual dynamic features of the data.[5] We show that, for a reasonable level of asymmetry, the pseudo-ML estimators of the four parameters of the skewed-t distribution are asymptotically normal even when the observations are generated by an α-stable distribution.[6] Consequently, the associated indirect inference estimators of the parameters of the α-stable distribution are asymptotically normal too.

We compare our method to two moment-based estimation methods. McCulloch (1986) proposed a quantile-based estimator, by building sample counterparts of the cumulative distribution function. Another approach is to match moments produced by the characteristic function (CF hereafter). Carrasco and Florens (2000, 2002) devise an optimal generalized method of moments based on a continuum of moment conditions corresponding to the CF computed at all points. Called continuous GMM (CGMM), the method produces an efficient estimator and overcomes the necessity of choosing an arbitrary set of frequencies, which was a fundamental drawback of CF-based methods.[7]

In a Monte Carlo study, we compare our estimator with CGMM and report that it is often more efficient in finite sample. Since DuMouchel (1973) provides a way to compute the efficiency bound in the *i.i.d.* case, we are able to measure the performance of our indirect inference estimator with the ML benchmark. At least in this *i.i.d.* setting, the efficiency loss appears mainly negligible given the finite sample improvement brought about by indirect inference. We also compare our method to the simple but inefficient quantile-based estimator of McCulloch (1986). Our estimates are close to those obtained with the quantile-based method. However, our estimators appear to have a much smaller variance, both asymptotically and in finite sample.

Many of the properties of stable models are shared by GARCH models. In particular, both models share the facts that the unconditional distribution has fat tails and that the tail shape is invariant under aggregation (see Ghose and Kroner, 1995; de Vries, 1991).[8] We illustrate this observational equivalence by generating different GARCH(1, 1) and IGARCH(1, 1) with Gaussian and Student-t innovations and aggregating the generated processes to lower frequencies. We show that the unconditional density captures very well the variance and kurtosis through aggregation and memory. The tail index α remains relatively constant under aggregation while the estimated dispersion increases.

We complete our analysis by applying our estimation procedure to a series of realized jumps filtered from the S&P 500 return series using the methodology of Tauchen and Zhou (2011). We find that the stable distribution that best characterizes these jumps is symmetric with an estimated tail index of 1.7.

The rest of the paper is organized as follows. Section 2 briefly describes the properties of α-stable distributions and their estimation by CGMM and empirical quantiles. In Section 3 we detail the application of the indirect inference methodology to the α-stable distribution, using the skewed-t distribution as an auxiliary model. We discuss the primitive conditions that warrant identification of structural parameters and asymptotic normality of their indirect inference estimators. Section 4 reports the results of a Monte Carlo study where indirect inference is compared to CGMM and empirical quantiles. The superior performance of CII is documented through both asymptotic and Monte Carlo MSE. We also compare and illustrate through simulations the relationship between the fat-tailed unconditional distributions produced by highly persistent GARCH models and an α-stable model. Section 5 is devoted to an empirical application to jumps in equity returns. Section 6 concludes. Proofs to several propositions are provided in the Appendix.

2. The α-stable distributions, CGMM and empirical quantiles

The α-stable family of distributions is characterized by four parameters α, β, σ and μ, where α is the stability parameter, β the skewness parameter, σ the scale parameter, and μ the location parameter. These parameters define the natural logarithm of the characteristic function as

$$\begin{aligned}\ln \psi_\theta(t) &= \ln E(\exp(it\,Y)) \\ &= i\mu t - \sigma^\alpha |t|^\alpha [1 - i\beta\, sign(t) \tan(\pi\alpha/2)]\end{aligned}$$

where $\theta = (\alpha, \beta, \sigma, \mu) \in \Theta =]1, 2] \times [-1, 1] \times \mathbb{R}_+^* \times \mathbb{R}$, Y is the random variable following the α-stable distribution $S(\theta)$ with characteristic function $\psi_\theta(\cdot)$ and $sign(t) = t/|t|$ for $t \neq 0$ (and 0 for $t = 0$). Note that the α-stable distribution can also be defined for α smaller than 1 but we preclude this case to guarantee the existence of a finite expectation. This requirement is rather realistic for the application to financial returns we have in mind. More generally, $E(|Y|^p) < \infty$ for all $p < \alpha$ and in particular $E(Y) = \mu$.

Even though the likelihood function is not known in closed form in general, the score function for an *i.i.d.* sample of size n remains asymptotically root-n normal. Therefore, DuMouchel (1973) was able to show that the standard tools of maximum likelihood theory (mainly root-n asymptotic normality and Cramer–Rao bounds) may be applied to estimation of θ insofar as its domain is limited to $|\beta| < \min(\alpha, 2 - \alpha)$. This result implies that efficient estimation of the parameters of α-stable distributions remains a sensible goal and that asymptotic normality of M-estimators like MLE or QMLE can be derived by the application of standard central limit theory to well-chosen (pseudo)-score functions rather than to moments of Y, which do not exist. This idea is the main motivation of the indirect inference strategy proposed in this paper.

Other estimation methods are available. Since the theoretical characteristic function has a closed form, estimation can be performed by fitting the sample characteristic function $n^{-1}\sum_{j=1}^{n}\exp(it_kY_j)$ to the theoretical one $\psi_\theta(t_k)$, defined on a grid of frequencies t_k, $k = 1, \ldots, K$. The problem is that it takes an infinite number of moment conditions, indexed by $t_k \in \mathbb{R}$, to summarize the informational content of the characteristic function. Consider the moment conditions:

$$E(h(t_k, Y, \theta)) = 0, \quad \forall k = 1, \ldots, K, \tag{1}$$

where $h(t_k, Y, \theta) = \exp(itY) - \psi_\theta(t_k)$. They amount to a set of $2K$ moment restrictions $E(g_k(\theta, Y)) = 0$ that include the real and imaginary parts of $h(t_k, Y, \theta)$. Standard GMM estimates are solutions of $\min ||\Omega_n^{-1/2} h_n(., Y, \theta)||$ where $h_n(., Y, \theta)$ is the sample

[5] The use of a wrongly specified simulator in indirect inference has not received much attention, except in Dridi et al. (2007).

[6] According to our Monte Carlo experiments, the allowed level of asymmetry is actually consistent with the one produced by an α-stable distribution with support on the whole real line.

[7] Some authors, like Fielitz and Rozelle (1981), recommend to match only a few frequencies on the basis of Monte Carlo results, while others, like Feuerverger and McDunnough (1981), recommend on the contrary to use as many frequencies as possible.

[8] It is well known that, except for the limiting case of the normal distribution, all the α-stable distributions have infinite variance. However, it should be remembered that a highly persistent GARCH with, by definition, finite conditional variances, may produce infinite moments at orders not much higher than two.