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An I(d) model with trend and cycles

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ABSTRACT

This paper deals with models allowing for trending processes and cyclical component with error processes that are possibly nonstationary, nonlinear, and non-Gaussian. Asymptotic confidence intervals for the trend, cyclical component, and memory parameters are obtained. The confidence intervals are applicable for a wide class of processes, exhibit good coverage accuracy, and are easy to implement.

1. Introduction

To start, consider the basic model

$$X_t = \beta_1 + \beta_2 t + u_t, \quad t = 1, 2, \dots, n, \tag{1.1}$$

where $\mathrm{E}\,(u_t) = 0$. When $\{u_t\}$ is a long memory stationary process, with memory parameter $d \in (-1/2, 1/2)$, the problem of estimating such models has been studied extensively in the literature. Yajima (1988, 1991) derived conditions for consistency and asymptotic normality of Least Squares (LS) estimators of the parameters of a regression model with nonstochastic regressors, when the errors $\{u_t\}$ have long memory. Dahlhaus (1995) suggested an efficient weighted least squares estimator for β_1 and β_2 and investigated its asymptotic properties in the case of a polynomial regression with stationary errors. Nonlinear regression models with long memory errors have been investigated by Ivanov and Leonenko (2004, 2008). The estimation of a trend when $\{u_t\}$ has $d \in [0, 3/2)$ was discussed by Deo and Hurvich (1998), but they did not estimate d and they required $\{u_t\}$ to have a linear structure with restrictive asymptotic weights.

There is a large literature on the estimation of d in the case of long memory. Fewer papers have so far considered an extended range for d to include regions of nonstationarity. Assuming that $\{u_t\}$ is observed, to estimate d, Velasco (1999, 2003) used data differencing and data tapering, and he noted that this inflates the estimator's variance. Robinson (2005) suggested an adaptive semiparametric estimation method for the case of a polynomial regression with fractionally-integrated errors, employing in his Monte Carlo study a tapered estimate of d. An alternative approach was developed by Shimotsu and Phillips (2005) who introduced an exact local Whittle estimation method based of fractional differencing of $\{u_t\}$, which is valid when a nonstationary process $\{u_t\}$ is generated by a linear process. Abadir et al. (2007) extended the classical Whittle estimator to the Fully-Extended Local Whittle (FELW) estimator that is valid for a wider range of d values, allowing for nonstationary $\{u_t\}$ but not for deterministic components.

The present papers focuses on the estimation of the linear regression model (1.1) and its extended version

$$X_t = \beta_1 + \beta_2 t + \beta_3 \sin(\omega t) + \beta_4 \cos(\omega t) + u_t, \quad t = 1, 2, \dots, n, \tag{1.2}$$

which allows for stationary and nonstationary errors $\{u_t\}$ and a cyclical component with $\omega \in (0, \pi)$. We assume that ω is known in (1.2), and so we treat separately the boundary case $\omega = 0$ as model (1.1), effectively covering $\omega \in [0, \pi)$ in the paper but not the

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unrealistic case of $\omega = \pi$ that leads to $X_t = \beta_1 + \beta_2 t + \beta_4 (-1)^t + u_t$. We do not propose a method of estimation for ω; see Nandi and Kundu (2003) and references therein for the estimation of ω in the context of a short memory linear process and no linear trend. Estimating ω is beyond the scope of this paper, though (as we will show) our procedure allows for time-varying ω and/or multiple cyclical components with different frequencies $\omega_\bullet$. For expository purposes, we refrain from writing these features into the model given in this introduction.

In this paper, we estimate $\boldsymbol{\beta} := (\beta_1, \beta_2, \beta_3, \beta_4)'$ by LS and generalize (for the presence of trend and cycle) the Fully-Extended Local Whittle (FELW) estimator of d given in Abadir et al. (2007). We also provide a simpler alternative form for the FELW estimator. We show that our estimators are consistent and we obtain their rates of convergence and limiting distributions, as well as confidence intervals based on them. The asymptotic properties of our LS estimators of β_1, β_2 turn out to be unaffected by (and robust to) the unknown cyclical component.

The papers listed earlier require the assumptions of linearity or Gaussianity of the error process. However, our estimation procedure allows for a wide range of permissible values of the memory parameter d and for possibly nonstationary, nonlinear, and nonnormal processes $\{u_t\}$. By virtue of $\{u_t\}$ being modelled semiparametrically, the procedure also allows for seasonality and other effects to be present in $\{u_t\}$ at nonzero spectral frequencies.

In Section 2, we investigate the LS estimators of $\beta_1, \beta_2, \beta_3, \beta_4$, while Section 3 is concerned with the parameters of the process $\{u_t\}$. Section 4 contains the results of simulation experiments on the performance of the estimators suggested earlier. It is technically straightforward to extend our results to higher-order polynomials and to values of d outside the interval $-1/2 < d < 3/2$ to which we restrict our attention in this paper. We do not report such extensions in order to simplify the exposition and because most economic series will not require more than a linear trend or d outside $-1/2 < d < 3/2$. The proofs of the main results are given in the Appendix.

We use $\xrightarrow{p}$ and $\xrightarrow{d}$ to denote convergence in probability and in distribution, respectively. We write i for the imaginary unit, 1_A for the indicator of a set A, $\lfloor \nu \rfloor$ for the integer part of ν, C for a generic constant but $c_\bullet$ for specific constants. The lag operator is denoted by L, such that $\mathrm{L}u_t = u_{t-1}$, and the backward difference operator by $\nabla := 1 - \mathrm{L}$. We define $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.

Definition 1.1. Let $d = k + d_\xi$, where $k = 0, 1, 2, \ldots$ and $d_\xi \in (-1/2, 1/2)$. We say that $\{u_t\}$ is an I(d) process (denoted by $u_t \sim \mathrm{I}(d)$) if

$$\nabla^k u_t = \xi_t, \quad t = 1, 2, \ldots,$$

where the generating process $\{\xi_t\}$ is a second order stationary sequence with spectral density

$$f_\xi(\lambda) = b_0 |\lambda|^{-2d_\xi} + o(|\lambda|^{-2d_\xi}), \quad \text{as } \lambda \to 0 \tag{1.3}$$

where $b_0 > 0$.

Notice that there are two parameters of interest in this definition, b_0 and d_ξ.

2. Estimation of $\boldsymbol{\beta}$

We will use Ordinary LS (OLS) estimation of $\boldsymbol{\beta}$, because of its ease of application, its consistency, and its asymptotic normality. Feasible Generalized LS (GLS) applied to (1.1) would require us to specify the autocovariance structure explicitly, which is not usually known, so OLS is more in line with the semiparametric approach of our paper. Even so, assuming the autocovariance structure is known and is correctly specified, it has been shown that the loss of efficiency will not be substantial in this context. For example, Table 1 of Yajima (1988) implies that the maximal loss of asymptotic efficiency by OLS compared to the BLUE is 11% when estimating β_1 and β_2, and 2% when estimating the mean of the differenced data (hence β_2 of the original data). These will correspond to our cases $d \in (-1/2, 1/2)$ and $d \in (1/2, 3/2)$, respectively, as will be seen later. These efficiency bounds apply to GLS as well, since it is a linear estimator, thus limiting the efficiency loss of OLS relative to GLS.

Below it will be shown that the rates of convergence of the OLS estimators depend on the order of integration d of u_t, and their limits depend on the long run variance s_ξ^2 of $\{\xi_t\}$ which needs to be estimated. Property (1.3) of the spectral density f_ξ implies that

$$\begin{aligned} s_\xi^2 &= \lim_{n\to\infty} \mathrm{E}\left(n^{-1/2-d_\xi} \sum_{t=1}^{n} \xi_t\right)^2 \\ &= \lim_{n\to\infty} n^{-1-2d_\xi} \int_{-\pi}^{\pi} \left(\frac{\sin(n\lambda/2)}{\sin(\lambda/2)}\right)^2 f_\xi(\lambda)\mathrm{d}\lambda \\ &= p(d_\xi) b_0, \end{aligned} \tag{2.1}$$

where b_0 is defined in (1.3) and

$$\begin{aligned} p(d) &:= \int_{-\infty}^{\infty} \left(\frac{\sin(\lambda/2)}{\lambda/2}\right)^2 |\lambda|^{-2d} \mathrm{d}\lambda \\ &= \begin{cases} 2\dfrac{\Gamma(1-2d)\sin(\pi d)}{d(1+2d)}, & \text{if } d \neq 0, \\ 2\pi, & \text{if } d = 0. \end{cases} \end{aligned}$$

To derive the asymptotic distribution of estimators of (β_1, β_2), we introduce the following condition on the generating process $\{\xi_t\}$ of Definition 1.1.

Assumption FDD. The finite-dimensional distributions of the process

$$Y_n(r) := n^{-1/2-d_\xi} \sum_{t=1}^{\lfloor nr \rfloor + 1} \xi_t, \quad 0 \le r \le 1 \tag{2.2}$$

converge to those of the Gaussian process $Y_\infty(r)$, that is,

$$Y_n(r) \xrightarrow{d} Y_\infty(r), \quad \text{as } n \to \infty.$$

Assumption FDD together with asymptotic (1.3) of spectral density f_ξ imply that

$$Y_\infty(r) = s_\xi J_{1/2+d_\xi}(r), \quad 0 \le r \le 1,$$

where $J_{1/2+d_\xi}(r)$ is a fractional Brownian motion. By definition, $J_{1/2+d_\xi}(r)$ is a Gaussian process with zero mean and covariance function

$$\begin{aligned} R_d(r, s) &:= \mathrm{E}\left(J_{1/2+d_\xi}(r) J_{1/2+d_\xi}(s)\right) \\ &= \frac{1}{2}\left(r^{1+2d_\xi} + s^{1+2d_\xi} - |r-s|^{1+2d_\xi}\right), \quad 0 \le r, s \le 1. \end{aligned} \tag{2.3}$$

2.1. Model (1.1)

In order to estimate the slope parameter β_2 and the location parameter β_1 of model (1.1), we use the OLS estimators

$$\widehat{\beta}_2 = \frac{\sum_{t=1}^{n} (X_t - \bar{X})(t - \bar{t})}{\sum_{t=1}^{n} (t - \bar{t})^2} \tag{2.4}$$

and

$$\widehat{\beta}_1 = \bar{X} - \widehat{\beta}_2 \bar{t}, \tag{2.5}$$