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# Topology optimization of continuum structures under buckling constraints

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ABSTRACT

This paper presents a study on topology optimization of continuum structures under buckling constraints. New algorithms are developed for minimization of structural compliance considering constraints on volume and buckling load factors. The SIMP (Solid Isotropic Material with Penalization) material model is employed and nodal relative densities are used as topology design variables. A new approach based on the eigenvalue shift and pseudo mode identification is proposed for eliminating the effect of pseudo buckling modes. Two-phase optimization algorithms are also proposed for achieving better optimized designs. Numerical examples are presented to illustrate the effectiveness of the new methods.



## 1. Introduction

Structural strength, stiffness, and stability are three of the important factors considered for assessing the design of a structure. Naturally, it is important to consider structural stability in the optimization process. Recently, buckling optimization has drawn more research attention.

For trusses, frames, and other built-up structures consisting of bars and beams, much work has been done to consider the stability requirements in structural optimization, such as size optimization of trusses and frames [1], shape optimization of columns, truss or built-up structures [2–4] and topology optimization of truss structures [5–7].

Neves et al. [8] and Min and Kikuchi [9] have considered structural stability in topology optimization of continuum structures. They investigated the reinforcement of a structure to increase its overall stability. Neves et al.[10] extended their earlier work to the buckling optimization of periodic material micro-structures. Geometrically nonlinear models have also been introduced into the optimization of continuum structures against buckling [11–15]. In addition, optimization of composite structures was considered by Lindgaard and Lund [16,17].

A common problem in topology optimization using the SIMP material model is the appearance of pseudo buckling modes in low-density regions. Neves et al. [8] suggested ignoring the geometrical stiffness matrices of elements with densities and principal stresses smaller than predefined threshold values. Meanwhile, they indicated that the predefined values might have a significant influence on optimization results. Bendsøe and Sigmund [18] pointed out that doing this might cause solution oscillations due to abrupt changes of objective functions and sensitivities. In order to avoid the discontinuity caused by such a cut-off method, they suggested the use of different penalization schemes for element stiffness matrix and geometric stiffness matrix. Currently this method appears to be a standard solution for this problem and has been used by many researchers, e.g. Lindgaard and Lund [19]. However, Zhou [20] showed that it might be difficult to select an appropriate parameter value for the expression of penalization in calculating accurate buckling load factors.

Pseudo eigenmodes may also appear in the optimization of eigenfrequencies in vibration problems [21]. To eliminate these pseudo modes, some methods of modifying element stiffness matrix and/or mass matrix in low-density regions have been proposed and details of these methods can be found in the research literature, e.g. [21–23]. A topology optimization problem considering buckling differs from the one considering vibration modes and is more complex as element geometrical stiffness matrices are dependent on element stresses, which depend both on the structure itself and on the loading condition. In contrast, element mass matrices used in frequency analysis are dependent on material distribution only.

In this paper, the pseudo buckling mode problem is investigated and a new method combining eigenvalue shift and pseudo mode identification is proposed. An optimization formulation for

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minimizing the structural compliance under material volume and buckling load factor constraints is used in the study.

This paper is organized as follows: In Section 2 the optimization formulation and material model used are presented. In Section 3, the finite element model employed for the structural analysis is introduced. In Section 4, expressions for the sensitivity of constraint functions and objective function are derived. In Section 5, some of the existing methods for dealing with pseudo buckling modes are briefly discussed and a new approach is proposed. In Section 6, new optimization algorithms are developed. In Section 7, two numerical examples are presented to demonstrate effectiveness of the proposed methods. Finally, concluding remarks are made.

## 2. Problem formulation and material interpolation scheme

### 2.1. Optimization problem formulation

The topology optimization of continuum structure may often generate designs with slender components when the allowed material volume fraction is small. If compressive stresses occur in these structural components, structural buckling may present serious safety concerns. Therefore, structural stability requirements should be considered in the optimization. The mathematical formulation of the compliance minimization problem of continuum structures with constraints on the material volume and buckling load factors can be stated as

$$\begin{aligned} &\text{find} \quad \boldsymbol{\rho} = \{\rho_1, \rho_2, \cdots, \rho_N\} \\ &\min \quad C = \mathbf{F}^{\mathrm{T}}\mathbf{U} = \mathbf{U}^{\mathrm{T}}\mathbf{K}\mathbf{U} \\ &\text{s.t.} \quad \mathbf{K}\mathbf{U} = \mathbf{F} \\ &\qquad \min_{j\in J}|\lambda_j| \geqslant \underline{\lambda} > 0 \\ &\qquad V(\boldsymbol{\rho}) \leqslant V_0 \\ &\qquad 0 < \underline{\rho} \leqslant \rho_i \leqslant 1 \qquad i = 1, 2, \ldots, N \end{aligned} \tag{1}$$

where $\rho_i$ $(i = 1, 2, \ldots, N)$ are design variables of relative material density; $N$ is the number of design variables; $C$ is the structural compliance; $\mathbf{U}$ and $\mathbf{F}$ are the global displacement and force vectors; $\lambda_j$ is the $j$th buckling load factor corresponding to the given load cases; $J$ is a set of indices of the buckling mode considered in the optimization; $\underline{\lambda}$ denotes the lower bound of buckling load factors; $V(\boldsymbol{\rho})$ is the total material volume of the structure; $V_0$ is the upper bound of material volume; and $\underline{\rho}$ is the lower bound of design variables, e.g. $\underline{\rho} = 0.001$.

Through the introduction of an explicit constraint condition on buckling load factors, designs that fail to satisfy stability requirements will be excluded from the feasible solution set. Theoretically, different levels of safety margins can be achieved by using different lower bound values. For example, if $\underline{\lambda} = 1$, the optimized structure will be at a critical state under normal service conditions; if $\underline{\lambda} > 1$, the structure will be stable under normal service conditions with a bigger safety margin for a bigger $\underline{\lambda}$; if $0 < \underline{\lambda} < 1$, the structure may buckle under normal service conditions, but cannot be a mechanism.

The buckling mode index set $J$ is introduced for two reasons. Firstly, when an applied load always points in the same direction, negative loading factors are meaningless and in this case, set $J$ should contain only the modes with positive load factors. Secondly, when pseudo modes are among the calculated buckling modes, the corresponding mode indices must be excluded from set $J$ as these modes are not real and should be ignored.

### 2.2. Material interpolation

It is possible to obtain continuous material distributions by using nodal relative densities as topology design variables [24]. It is noted that Kang and Wang [25] have presented a more general density interpolation strategy for topology optimization using nodal design variables and Shepard interpolation. In this study, a more conventional interpolation scheme based on element nodal values and shape functions is used. Within the $e$th element, the relative density distribution is expressed as

$$\rho^e(x, y) = \sum_{k=1}^{NN} N_k(x, y)\rho_k^e \tag{2}$$

where $\rho_k^e$ denotes nodal density value at the $k$th node of the element, $NN$ is the number of nodes in the element, and $N_k(x, y)$ is the element shape function for the $k$th node.

Using the SIMP material model, the elasticity matrix at point $(x, y)$ is expressed in terms of material relative density $\rho^e(x, y)$

$$\mathbf{E}(x, y) = [\rho^e(x, y)]^p \mathbf{E}_0 \tag{3}$$

where $\mathbf{E}_0$ is the elasticity matrix of the isotropic solid elastic material, and $p \geqslant 1$ is a penalization exponent number.

## 3. Finite element analysis methods

In this section, the finite element model for structural analyses and the computation of buckling load factors using hybrid stress element is briefly introduced.

### 3.1. Finite element model

When the nodal design variable is employed, the checkerboard patterns can be avoided naturally. However, a "layering" or "islanding" phenomenon of black and white regions in the design domain may appear [24]. Deng et al. [26] showed that this problem could be effectively avoided by replacing the conventional four-node displacement-based quadrilateral element with a hybrid stress element. The same approach is taken in this study, and in this section, the basic theory and formulation of the hybrid stress element to be used will be summarized.

Pian and Sumihara [27] developed a four-node hybrid stress finite element for homogeneous plane problems. Independent element stress and displacement fields are defined and can be expressed as

$$\boldsymbol{\sigma} = \left\{\sigma_x, \sigma_y, \tau_{xy}\right\}^{\mathrm{T}} = \boldsymbol{\Phi}\boldsymbol{\beta} \tag{4}$$

$$\mathbf{u} = \left\{u_x, u_y\right\}^{\mathrm{T}} = \mathbf{N}\mathbf{d} \tag{5}$$

where $\sigma_x, \sigma_y$ and $\tau_{xy}$ are stress components, $u_x$ and $u_y$ are displacement components, $\boldsymbol{\Phi}$ and $\mathbf{N}$ are interpolation matrices for element stress and displacement fields, respectively, $\boldsymbol{\beta}$ is an element stress parameter vector, and $\mathbf{d}$ is the nodal displacement vector.

Based on the Hellinger–Reissner variational principle, the following expressions for element stiffness matrix $\mathbf{K}_e$ and the stress parameter vector $\boldsymbol{\beta}$ can be derived

$$\mathbf{K}_e = \mathbf{G}_e^{\mathrm{T}}\mathbf{H}_e^{-1}\mathbf{G}_e \tag{6}$$

$$\boldsymbol{\beta} = \mathbf{H}_e^{-1}\mathbf{G}_e\mathbf{d} \tag{7}$$

where matrices $\mathbf{G}_e$ and $\mathbf{H}_e$ are defined as

$$\mathbf{G}_e = \int_{-1}^{1}\int_{-1}^{1}\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{B}|\mathbf{J}|\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{8}$$

$$\mathbf{H}_e = \int_{-1}^{1}\int_{-1}^{1}\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{S}_0\boldsymbol{\Phi}\frac{|\mathbf{J}|}{[\rho^e(\xi, \eta)]^p t_0}\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{9}$$