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Infinite-dimensional VARs and factor models[☆]

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ABSTRACT

This paper proposes a novel approach for dealing with the 'curse of dimensionality' in the case of infinite-dimensional vector autoregressive (IVAR) models. It is assumed that each unit or variable in the IVAR is related to a small number of neighbors and a large number of non-neighbors. The neighborhood effects are fixed and do not change with the number of units (N), but the coefficients of non-neighboring units are restricted to vanish in the limit as N tends to infinity. Problems of estimation and inference in a stationary IVAR model with an unknown number of unobserved common factors are investigated. A cross-section augmented least-squares (CALS) estimator is proposed and its asymptotic distribution is derived. Satisfactory small-sample properties are documented by Monte Carlo experiments. An empirical illustration shows the statistical significance of dynamic spillover effects in modeling of US real house prices across the neighboring states.

1. Introduction

Vector autoregressive (VAR) models provide a flexible framework for the analysis of complex dynamics and interactions that exist across economic variables or units. Traditional VARs assume that the number of such variables, N, is fixed and the time dimension, T, tends to infinity. But since the number of parameters to be estimated grows at a quadratic rate with N, in practice the empirical applications of VARs often involve only a handful of variables. The objective of this paper is to consider VARs where *both* N and T are large. In this case, parameters of the VAR model can no longer be consistently estimated unless suitable restrictions are imposed to overcome the dimensionality problem.

Two different approaches have been suggested in the literature to deal with this 'curse of dimensionality': (i) shrinkage of the parameter space, and (ii) shrinkage of the data. Spatial and/or spatiotemporal literature shrinks the parameter space by using *a priori* given spatial weight matrices that restrict the nature of the links across the units. Alternatively, prior probability distributions are imposed on the parameters of the VAR such as the 'Minnesota' priors proposed by Doan et al. (1984). This class of models is known as Bayesian VARs (BVARs).[1]

The second approach is to shrink the data, along the lines of index models. Geweke (1977) and Sargent and Sims (1977) introduced dynamic factor models, which have more recently been generalized to allow for weak cross-section dependence by Forni and Lippi (2001) and Forni et al. (2000, 2004). Empirical evidence suggests that few dynamic factors are needed to explain

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1 Other types of prior have also been considered in the literature. See, for example, Del Negro and Schorfheide (2004) for a recent reference. In most applications, BVARs have been applied to relatively small systems (e.g. Leeper et al. (1996) considered 13- and 18-variable BVARs; a few exceptions include Giacomini and White (2006) and De Mol et al. (2008)), with the focus being mainly on forecasting. Bayesian VARs are known to produce better forecasts than unrestricted VARs or structural models. See Litterman (1986) and Canova (1995) for further references.

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the co-movements of macroeconomic variables.[2] This has led to the development of factor-augmented VAR (FAVAR) models by Bernanke et al. (2005) and Stock and Watson (2005), among others.

Applied researchers are often forced to impose arbitrary restrictions on the coefficients that link the variables of a given cross-section unit to the current and lagged values of the remaining units, mostly because they realize that without such restrictions the model is not estimable. This paper proposes a novel way to deal with the curse of dimensionality by shrinking part of the parameter space in *the limit* as the number of variables (N) tends to infinity. An important example would be a VAR model in which each unit is related to a small number of neighbors and a large number of non-neighbors. The neighbors could be individual units or, more generally, linear combinations of units (spatial averages). The neighborhood effects are fixed and do not change with N, but the coefficients corresponding to the remaining non-neighbor units are small, of order $O(N^{-1})$. Such neighborhood and non-neighborhood effects could be motivated by theoretical economic considerations, or could arise due to the mis-specification of spatial weights.

Although under this set-up each of the non-neighboring coefficients is small, the sum of their absolute values in general does not tend to zero and the aggregate spatiotemporal non-neighborhood effects could be large. This paper shows that, under weak cross-section dependence, the spillover effects from non-neighboring units are neither particularly important, nor estimable.[3] But the coefficients associated with the neighboring units can be consistently estimated by simply ignoring the non-neighborhood effects that are of second-order importance in N. On the other hand, if the units are cross-sectionally strongly dependent, then the spillover effects from non-neighbors are in general important, and ignoring such effects can lead to inconsistent estimates.

Another model of interest arises when, in addition to the neighborhood effects, there is also a fixed number of dominant units that have non-negligible effects on all other units. In this case the limiting outcome is shown to be a dynamic factor model.[4] Accordingly, this paper provides a link between data and parameter shrinkage approaches to mitigating the curse of dimensionality. By imposing limiting restrictions on some of the parameters of the VAR we effectively end up with a data shrinkage. To distinguish high-dimensional VAR models from the standard specifications, we refer to the former as the infinite-dimensional VARs or IVARs for short.

The paper also establishes the conditions under which the global VAR (GVAR) approach proposed by Pesaran et al. (2004) is applicable.[5] In particular, the IVAR featuring all macroeconomic variables could be arbitrarily well approximated by a set of finite-dimensional small-scale models that can be consistently estimated separately in the spirit of the GVAR.

A second contribution of this paper is the development of appropriate econometric techniques for estimation and inference in stationary IVAR models with an unknown number of unobserved common factors. This extends the analysis of Pesaran (2006) to dynamic models where all variables are determined endogenously. A simple cross-section augmented least-squares estimator (or CALS for short) is proposed and its asymptotic distribution derived. Small-sample properties of the proposed estimator are investigated through Monte Carlo experiments. As an illustration of the proposed approach we consider an extension of the empirical analysis of real house prices across the 49 US states conducted recently by Holly et al. (2010), and show statistically significant *dynamic* spillover effects of real house prices across the neighboring states.

The remainder of the paper is organized as follows. Section 2 introduces the IVAR model. Section 3 investigates cross-section dependence in IVAR models. Section 4 focusses on the estimation of a stationary IVAR model. Section 5 discusses the results of the Monte Carlo (MC) experiments, and Section 6 presents the empirical results. The final section offers some concluding remarks. Proofs are provided in the Appendix.

We give a brief word on notation. $|\lambda_1(\mathbf{A})| \geq |\lambda_2(\mathbf{A})| \geq \cdots \geq |\lambda_n(\mathbf{A})|$ are the eigenvalues of $\mathbf{A} \in \mathbb{M}^{n\times n}$, where $\mathbb{M}^{n\times n}$ is the space of real-valued $n \times n$ matrices. $\|\mathbf{A}\|_1 \equiv \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|$ denotes the maximum absolute column sum matrix norm of $\mathbf{A}$, and $\|\mathbf{A}\|_\infty \equiv \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|$ is the absolute row sum matrix norm of $\mathbf{A}$. $\|\mathbf{A}\| = \sqrt{\varrho(\mathbf{A}'\mathbf{A})}$ is the spectral norm of $\mathbf{A}$, and $\varrho(\mathbf{A}) \equiv \max_{1\leq i\leq n}\{|\lambda_i(\mathbf{A})|\}$ is the spectral radius of $\mathbf{A}$.[6] All vectors are column vectors, and the ith row of $\mathbf{A}$ is denoted by $\mathbf{a}_i'$. $a_n = O(b_n)$ denotes that the deterministic sequence $\{a_n\}$ is at most of order b_n. $x_n = O_p(y_n)$ states that the random variable x_n is at most of order y_n in probability. $\mathbb{N}$ is the set of natural numbers, and $\mathbb{Z}$ is the set of integers. We use K and ϵ to denote positive fixed constants that do not vary with N or T. Convergence in distribution and convergence in probability is denoted by $\xrightarrow{d}$ and $\xrightarrow{p}$, respectively. Symbol $\xrightarrow{q.m.}$ represents convergence in quadratic mean. $(N, T) \xrightarrow{j} \infty$ denotes joint asymptotic in N and T, with N and $T \to \infty$, in no particular order.

2. Infinite-dimensional vector autoregressive models

Suppose we have T time series observations on N cross-section units indexed by $i \in \mathcal{S}_{(N)} \equiv \{1, \ldots, N\} \subseteq \mathbb{N}$. Individual units could be households, firms, regions, or countries. Both dimensions, N and T, are assumed to be large. For each point in time, t, and for each $N \in \mathbb{N}$, the N cross-section observations are collected in the $N \times 1$ vector $\mathbf{x}_{(N),t} = (x_{(N),1t}, \ldots, x_{(N),Nt})'$, and it is assumed that $\mathbf{x}_{(N),t}$ follows the VAR(1) model:

$$\mathbf{x}_{(N),t} = \boldsymbol{\Phi}_{(N)}\mathbf{x}_{(N),t-1} + \mathbf{u}_{(N),t}, \tag{1}$$

$$\mathbf{u}_{(N),t} = \mathbf{R}_{(N)}\boldsymbol{\varepsilon}_{(N),t}. \tag{2}$$

$\boldsymbol{\Phi}_{(N)}$ and $\mathbf{R}_{(N)}$ are $N \times N$ coefficient matrices that capture the dynamic and contemporaneous dependences across the N units, and $\boldsymbol{\varepsilon}_{(N),t} = (\varepsilon_{1t}, \varepsilon_{2t}, \ldots, \varepsilon_{Nt})'$ is an $N \times 1$ vector of white noise errors with mean $\mathbf{0}$ and covariance matrix $\mathbf{I}_N$.

VAR models have been extensively studied when N is small and fixed, and T is large and unbounded. This framework, however, is not appropriate for many empirical applications of interest. This paper aims to fill this gap by analyzing VAR models where both N and T are large. The sequence of models (1) and (2) with $\dim(\mathbf{x}_{(N),t}) = N \to \infty$ will be referred to as the infinite-dimensional VAR model, or IVAR for short. The extension of the

2 Stock and Watson (1999, 2002), Giannone et al. (2005) conclude that only a few, perhaps two, factors explain much of the predictable variations, while Bai and Ng (2007) estimate four factors and Stock and Watson (2005) estimate as many as seven factors.

3 Concepts of strong and weak cross-section dependence, introduced in Chudik et al. (2010), will be applied to VAR models.

4 The case of IVAR models with a dominant unit is studied in Pesaran and Chudik (2010).

5 The GVAR model has been used to analyse credit risk in Pesaran et al. (2006, 2007). An extended and updated version of the GVAR by Dées et al. (2007), which treats the Euro area as a single economic area, was used by Pesaran et al. (2007) to evaluate UK entry into the Euro. Global dominance of the US economy in a GVAR model is considered in Chudik (2008). Further developments of a global modeling approach are provided in Pesaran and Smith (2006). Garratt et al. (2006) provide a textbook treatment of GVAR.

6 Note that, if $\mathbf{x}$ is a vector, then $\|\mathbf{x}\| = \sqrt{\varrho(\mathbf{x}'\mathbf{x})} = \sqrt{\mathbf{x}'\mathbf{x}}$ corresponds to the Euclidean length of vector $\mathbf{x}$.