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Nonlinear modal analysis of nonconservative systems: Extension of the periodic motion concept

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ABSTRACT

As the motions of nonconservative autonomous systems are typically not periodic, the definition of nonlinear modes as periodic motions cannot be applied in the classical sense. In this paper, it is proposed to 'make the motions periodic' by introducing an additional damping term of appropriate sign and magnitude. It is shown that this generalized definition is particularly suited to reflect the periodic vibration behavior induced by harmonic external forcing or negative linear damping. In a large range, the energy dependence of modal frequency, damping ratio and stability is reproduced well. The limitation to isolated or weakly-damped modes is discussed.

1. Introduction

The concept of nonlinear modes is an attempt to extend the ideas of linear modal analysis to nonlinear systems. Nonlinear modal analysis is useful to characterize and quantify the energy dependence of the essential vibration behavior of nonlinear systems. This facilitates the understanding of phenomena like energy transfer, localization and modal interactions that may be caused by nonlinear effects. Moreover, nonlinear modes can be used for the purpose of model reduction. To this end, the problem is reduced to the typically two-dimensional subspace spanned by a specific nonlinear mode. This can greatly improve the computational efficiency of vibration predictions, which is crucial e.g. for design optimization and uncertainty analysis involving detailed models of nonlinear structures [1,2]. For recent overviews on the general topic, the interested reader is referred to [3,4].

Useful mathematical properties are lost when nonlinear effects become important. For this reason, the definition of nonlinear modes[1] is less straight forward. There are basically two different established definitions: (a) the periodic motion definition and (b) the invariant manifold definition.

According to definition (a), nonlinear modes are viewed as periodic motions of the autonomous nonlinear system [5,6]. A family of periodic motions can be defined that are continuations with respect to the kinetic energy. These branches of periodic solutions of the equation of motion may or may not be connected to a corresponding linear mode at low energies. By using continuation and bifurcation analysis, interactions between different nonlinear modes can be resolved. The periodic motion definition directly associates properties of the flow to the modes such as its frequency and its asymptotic stability. Unfortunately, it does not apply to nonconservative systems, since their autonomous motions are typically only periodic at possible limit cycles, i.e. at specific energy values. Being inspired by the modes of damped linear systems, Laxalde and Thouverez [7] defined nonlinear modes as pseudo-periodic motions. They computed them by means of the generalized Fourier–Galerkin method. In contrast to the oscillatory term in the usual Fourier ansatz, they considered an additional decay term. This approach is, however, strictly limited to trigonometric base functions, and the notion of mode stability has not been established yet.

According to definition (b), nonlinear modes are viewed as an invariant relationship between the coordinates of an autonomous system. This relationship defines a manifold in the system's phase space that includes the equilibrium point, where it is tangential to the hyper-plane spanning the locus of the corresponding linear mode [8–10]. In the simplest case without internal resonances, this manifold is two-dimensional. Hence, a point on the manifold can be

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[1] In literature, the term Nonlinear Normal Mode (NNM) is quite common. However, the term 'normal' may mislead to the wrong conclusion that nonlinear modes are orthogonal to each other. Apparently this term goes back to Rosenberg [5], who defined nonlinear modes as vibrations in *unison*, i.e. where all material points cross their equilibrium point and their extremum points simultaneously. For this type of vibration, the motions take place on so-called modal lines in the generalized displacement space which are normal to the surface of maximum potential energy [4]. However, this property is only valid for symmetric conservative systems. Hence the term 'normal' in this context is avoided throughout this article.

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parameterized uniquely by a suitable set of two coordinates. Only the geometry of the invariant manifold is governed by this definition. To assess the vibratory properties such as frequency and stability, the in-manifold flow can be analyzed in a second step. Conceptually, the invariant manifold definition is not limited to conservative systems. Compared to periodic motion based methods, two important difficulties are often reported in practice: the difficulty to find a suitable set of coordinates for a unique parametrization and the computational burden. The former difficulty often arises in conjunction with localization and modal interactions at higher energies. These phenomena can lead to a folding of the invariant manifold in certain coordinate systems, so that not every point on it can be described uniquely anymore, see e.g. [11,12]. In terms of computational effort, it can generally be stated that periodic motion based procedures tend to be more efficient than invariant manifold based ones. For periodic motion based methods, only a single periodic orbit needs to be computed at the same time. In contrast, often the entire manifold, being the locus of infinitely many orbits, has to be determined simultaneously in the case of invariant manifold based methods. This results in a larger problem dimension and can produce considerable computational effort. To simplify the problem, the manifold can be divided into annular regions of finite size. But the unsteady character of the flow of nonconservative systems generally leads to a coupling of these regions, which requires special attention [12].

The purpose of this article is to extend the periodic motion definition to dissipative systems. The general concept is introduced in Section 2. Two computational implementations of the extended concept are outlined in Section 3. They are based on Shooting and Harmonic Balance, respectively. In Section 4, the capabilities and limitations of the approach are assessed for several numerical examples.

2. Extension of the periodic motion concept to dissipative systems

This section is divided into three subsections: First, the motivation and general idea for the extension of the periodic motion concept is presented. Then, the nonlinear modes are defined in such a way that they are capable of characterizing this dynamic regime in terms of eigenfrequency, modal damping ratio and vibrational deflection shape. Finally, analogies to the method of force appropriation are indicated.

2.1. Periodic vs. damped motion concept

Suppose the motions of an autonomous nonlinear system are described by a finite set of generalized coordinates $\mathbf{u}$ and velocities $\dot{\mathbf{u}}$, governed by a set of second-order ordinary differential equations

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{f}(\mathbf{u}(t), \dot{\mathbf{u}}(t)) = \mathbf{0}. \tag{1}$$

Herein, $\mathbf{M}$ is the mass matrix and $\mathbf{f}$ are linear or nonlinear restoring as well as dissipative forces.

In the conservative case, the nonlinear modes represent periodic solutions to Eq. (1). Once the periodic motion is initiated, the autonomous system will retain this motion for all times. This motion can be initiated by according initial conditions. Also, this motion can be induced by harmonic external forcing at resonance, when an appropriate term is included into the autonomous Eq. (1). In other words, the frequency-energy characteristic of the nonlinear mode is identical to the backbone of the frequency response curves for varying excitation level. This is known as the 'deformation-at-resonance' property of nonlinear modes [13,3].

In the presence of nonconservative forces, an unsteady motion takes place instead until an attractor is reached, for instance a limit cycle or an equilibrium point. If the system is subjected to a harmonic external forcing at resonance, this behavior is changed and a periodic motion may persist instead. Depending on the excitation level, different kinetic energy levels can be reached. In the nonconservative case, the deformation-at-resonance property does not hold anymore, i.e. the backbone curve of the (steady-state) frequency response curves is not identical to the (instantaneous) frequency-energy characteristic of the autonomous system. In fact, this deviation is not due to nonlinearity, but also holds in the linear case: Consider the behavior of a single-degree-of-freedom oscillator with undamped eigenfrequency ω_0 and damping ratio D. While the backbone curve is a straight vertical line through the constant resonance frequency $\omega_{\mathrm{res}} = \omega_0\sqrt{1-2D^2}$, the frequency of the autonomous system is the damped eigenfrequency $\omega_{\mathrm{d}} = \omega_0\sqrt{1-D^2}$.

Owing to this peculiarity of nonconservative systems, it is not ad hoc clear how to define the nonlinear modes. Essentially, there are two different opportunities:

- *damped motion concept*: nonlinear modes shall capture the strict autonomous behavior, i.e. unsteady motions in general, or
- *periodic motion concept*: nonlinear modes are still periodic motions, which share, as much as possible, the properties of the underlying autonomous system.

Both conceptions are illustrated in Fig. 1. The former concept is the classical one and was followed for instance in [12,7]. In this study, the latter concept was followed. More specifically, periodic motions in the presence of (a) harmonic external forcing at resonance, and (b) self-excitation by negative modal damping represent the dynamic regime of interest. The aim is therefore to capture the vibration behavior in situations where a (permanent) *resonant excitation source* is present. The term 'resonant' refers to the fact that in both cases, the oscillation frequency equals one of the system's (energy-dependent) eigenfrequencies. It is believed that this concept is more adjusted to persistent vibrations of nonconservative systems. Such vibrations are often of primary concern from a structural dynamic design point of view.

2.2. A new definition of nonlinear modes

The remaining question is now how to *make* the motions periodic. It is proposed to enforce periodicity by an additional damping term $\xi\mathbf{M}\dot{\mathbf{u}}$ that is just large enough to compensate the nonconservative forces,

$$\mathbf{M}\ddot{\mathbf{u}}(t) - \xi\mathbf{M}\dot{\mathbf{u}}(t) + \mathbf{f}(\mathbf{u}(t), \dot{\mathbf{u}}(t)) = \mathbf{0}, \quad 0 \leqslant t < T \wedge \mathbf{u}(t+T) = \mathbf{u}(t). \tag{2}$$

The extended definition of nonlinear modes is therefore as follows: *A nonlinear mode is as a family of periodic motions of an autonomous nonlinear system. If the system is nonconservative, these periodic motions are enforced by mass-proportional damping/self-excitation.*

The choice of a mass-proportional damping ensures consistency with linear modal analysis for the case of symmetric systems with modal damping. In this case, the nonlinear modes are orthogonal with respect to the mass matrix. Hence, the damping term does not affect the mode shape or the natural frequency ω_0. The damping term is related to the modal damping by $2D\omega_0 = \xi$. In the nonlinear regime, however, the modes are no longer orthogonal, so that the additional damping term can induce artificial modal coupling. This makes the approach intrusive. Owing to this possible source of inaccuracy, the valid dynamic regime will be restricted to *isolated modes* where strong modal interactions are absent. It