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# Profile quasi-maximum likelihood estimation of partially linear spatial autoregressive models[☆]

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ABSTRACT

We propose profile quasi-maximum likelihood estimation of spatial autoregressive models that are partially linear. The rate of convergence of the spatial parameter estimator depends on some general features of the spatial weight matrix of the model. The estimators of other finite-dimensional parameters in the model have the regular $\sqrt{n}$-rate of convergence and the estimator of the nonparametric component is consistent but with different restrictions on the choice of bandwidth parameter associated with different natures of the spatial weights. Monte Carlo simulations verify our theory and indicate that our estimators perform reasonably well in finite samples.



## 1. Introduction

Since Paelinck coined the term "spatial econometrics" in the early 1970s to refer to a set of methods that explicitly deals with spatial dependence and spatial heterogeneity, the field has grown rapidly. The books by Cliff and Ord (1973), Paelinck and Klaassen (1979), Anselin (1988), Cressie (1993) and Anselin and Florax (1995) contribute significantly to the development of the field. For a recent survey on the subject, see Anselin and Bera (2002).

Among the class of spatial models, spatial autoregressive (SAR) models on lattices have attracted huge attention. Various methods have been proposed to estimate the SAR models, which include the method of maximum likelihood (ML) by Ord (1975) and Smirnov and Anselin (2001), the method of moments (MM) by Kelejian and Prucha (1999, 2010), and the method of quasi-maximum likelihood estimation (QMLE) by Lee (2002b, 2004). A common feature of these methods is that they are all developed to estimate finite dimensional parameters in the SAR models which are frequently assumed to be linear. When an unknown infinite dimensional parameter is present (e.g., the regression function is of unknown form), there is a lack of guidance on the estimation and inference process.

In this paper, we consider spatial autoregressive (SAR) models on lattices when the regression function is partially specified, motivated by the following considerations. First, as was argued in Paelinck and Klaassen (1979, pp. 6–9), econometric relations in space result more often than not in highly non-linear specifications. It has well been documented in the literature that many economic variables exhibit nonlinear relationships. For example, economic inequality is associated with economic growth through an inverse-U shaped Kuznets curve. Recent study also suggests an inverse-U relationship between economic growth and environmental quality even when the spatial effect is accounted for (see Rupasingha et al., 2004). Ignoring the potential nonlinear relationship in spatial dependence models often results in inconsistent estimation of the parameters of interest and misleading conclusions.

Second, while most econometric analysis and empirical studies using the SAR models ignore potential nonlinear functional forms, there have been some considerations of flexible functional forms in the literature that try to take into account certain form of nonlinearities in the models. See, for example, van Gastel and Paelinck (1995), Baltagi and Li (2001), Pace et al. (2004) and Yang et al. (2006). Most of these papers introduce a parametric transformation (e.g., Box–cox transformation) on the response variable or/and

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regressors. Nevertheless, parametric functional form transformation can at most provide certain protection against some specific nonlinear forms. In the absence of a priori information and theoretical foundation, it is generally advisable to consider more flexible functional forms.

Third, as nonparametric techniques advance, more and more researchers find out the advantage of nonparametric and semiparametric methods in modeling nonlinear economic relationships (see, e.g., Yatchew, 1998). Recent researches have started addressing the importance of nonparametric modeling in spatial econometrics. For example, in modeling hedonic housing price, Gress (2004) introduced two semiparametric spatial autocorrelation models and compared them with a variety of competing parametric spatial models. He found that the semiparametric models offer more accurate and stable estimates of the regression parameters and better out-of-sample predictions than do the alternative parametric models. Basile and Gress (2004) proposed a semiparametric spatial auto-covariance specification of the growth model for the European economy and found that nonlinearities are important in regional growth in Europe even when the spatial dependence is controlled for. As a result, assuming a common linear relationship between economic growth and inputs is misleading.

Fourth, as Robinson (1988) remarked, a correctly specified parametric model affords precise inferences, a badly misspecified one, possibly seriously misleading ones, whereas nonparametric modeling is associated with both greater robustness and lesser precision. So an intermediate strategy is to apply a semiparametric form, among which partially linear models are widely used.

In this paper we extend the work of Lee (2004) and consider estimating the parameters in partially linear SAR models by the profile QMLE method. When the error term has a known density form, Staniswalis (1989) proposed estimating the nonparametric regression function by maximizing the local log-likelihood. In the case of unknown error density, we can apply the idea of the quasi-maximum likelihood (QML). Because we have both parametric and nonparametric components in our regression function, we first concentrate out the nonparametric component by expressing the nonparametric component as certain function of the parametric component and the data. Then we estimate the parametric component and recover the nonparametric component after that. Consequently, we term our estimator as a profile QML estimator. Like Lee (2004), our parametric component includes the spatial parameter, the coefficient of the linear part of the regression function, and the variance of the error term. Because the parametric component is of finite dimension, it is also called the finite dimensional parameter in the literature.

Like Lee (2004), the rates of convergence of the estimators for the finite dimensional parameters depend on some general features of the spatial weights matrix of the model. The estimator of the spatial parameter may indeed have a $\sqrt{n}$-rate of convergence and a normal limiting distribution. Nevertheless, under some circumstances, the estimator has a slow rate of convergence for some parametric components of the model, say when all elements of the spatial weights matrix tend to zero as the sample size goes to infinity. In the former case, the nonparametric component can be estimated consistently at the conventional nonparametric convergence rate. But this is not true in the latter case where more stringent conditions on the spatial weights matrix and the bandwidth parameter are required to gain consistency of the estimators for both the parametric and nonparametric components.

It is worth mentioning that the semiparametric models of Gress (2004) and Basile and Gress (2004) are special cases of our model. We can also apply our model to examine many other well-known nonlinear relationships in economics, including the relationship between economic inequality and economic growth, the relationship between economic growth and environmental inequality, the relationship between education and wages, etc.

The paper is structured as follows. In Section 2 we introduce the partially linear SAR model and the profile QMLE approach to estimate the finite and infinite dimensional parameters in the model. We make a set of assumptions in Section 3. In Section 4 we study the asymptotic properties of the profile QMLE estimators when the information matrix is nonsingular and the parametric component can be estimated at the regular $\sqrt{n}$-rate. In Section 5 we study the asymptotic properties of the profile QMLE estimators when the information matrix is singular and some of the parametric component can only be estimated at a slower rate. We conduct Monte Carlo simulations to check the performance of the proposed estimator in Section 6. Final remarks are contained in Section 7. All technical details are relegated to the Appendix.

Like Kelejian and Prucha (2001), we adopt the following notation and conventions. For a matrix $A_n$, we denote its norm as $\|A_n\| = [\mathrm{tr}(A_nA_n')]^{1/2}$ and the $(i,j)$th element of $A_n$ as $a_{n,ij}$. Similarly, for a vector $a_n$, $a_{n,i}$ denotes its $i$th element. An analogous convention is adopted for matrices and vectors that do not depend on the index $n$, where $n$ is frequently suppressed. We say $A_n$ is uniformly bounded in absolute value if $\sup_{1\le i\le n,1\le j\le n}|a_{n,ij}| < \infty$. We say $A_n$ is uniformly bounded in row sums (resp. column sums) if $\sup_{1\le i\le n}\sum_{j=1}^{n}|a_{n,ij}| \le c_a < \infty$ (resp. $\sup_{1\le j\le n}\sum_{i=1}^{n}|a_{n,ij}| \le c_a < \infty$).

## 2. Partially linear spatial autoregressive models and profile QMLE

In this paper we investigate estimation of the spatial autoregressive models:

$$Y_n = X_n\beta_0 + \mathbf{m}_0(Z_n) + \rho_0 W_n Y_n + U_n, \tag{2.1}$$

where $X_n \equiv (x_{n,1},\ldots,x_{n,n})'$ and $Z_n \equiv (z_{n,1},\ldots,z_{n,n})'$ are $n\times p$ and $n\times q$ matrices of regressors, respectively, $W_n$ is a specified constant $n\times n$ spatial weight matrix, $U_n \equiv (u_1,\ldots,u_n)'$ is an $n$-dimensional vector of i.i.d. disturbances with zero mean and finite variance $\sigma_0^2$, $\mathbf{m}_0(Z_n) \equiv (m_0(z_{n,1}),\ldots,m_0(z_{n,n}))'$, and $m_0(\cdot)$ is an unknown function defined on $\mathbb{R}^q$. Let $\theta_0 = (\beta_0', \rho_0, \sigma_0^2)'$ be the true finite dimensional parameter vector. Denote $T_n(\rho) = I_n - \rho W_n$ for any value of $\rho$. It follows that

$$Y_n = T_n^{-1}\left(X_n\beta_0 + \mathbf{m}_0(Z_n) + U_n\right), \tag{2.2}$$

provided $T_n \equiv T_n(\rho_0)$ is nonsingular.

Let $U_n(\delta) = Y_n - X_n\beta - \mathbf{m}_0(Z_n) - \rho W_n Y_n$, where $\delta = (\beta', \rho)'$. In the case for which $\mathbf{m}_0(\cdot)$ is missing from the definition of $U_n(\delta)$, Lee (2002b, 2004) proposes maximizing the Gaussian quasi log likelihood

$$\log L_n(\theta) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\sigma^2 + \log|T_n(\rho)| - \frac{1}{2\sigma^2}U_n(\delta)'U_n(\delta), \tag{2.3}$$

where $\theta = (\beta', \rho, \sigma^2)'$.

Since $\mathbf{m}_0(\cdot)$ is present in Eq. (2.1), we propose estimating $\theta$ by the following two step procedure: (i) Estimate $m_0(z)$ for fixed $\theta$, denote the resulting estimator as $m_\theta(z)$; (ii) Plug in $m_\theta(z)$ into $U_n(\delta)$ in (2.3), and obtain the QMLE estimator $\widehat{\theta}$ for $\theta$ and $m_{\widehat{\theta}}(z)$ for $m_0(z)$.

To estimate $m_0(z)$ for fixed $\theta$ in the first step, we generalize the approach of Staniswalis (1989) for likelihood-based estimation and use a method that might be called profile quasi-maximum likelihood estimation (QMLE). We give an asymptotic analysis based on the local polynomial procedure. See Fan (1992) and Fan and Gijbels (1996) for a discussion on the attractive properties of local polynomials.

Let $K(\cdot)$ denote a kernel function on $\mathbb{R}^q$ and $h = h_n$ a bandwidth sequence. Set $K_h(z) = h^{-q}K(z/h)$. Let $Y_n^*(\rho) = T_n(\rho)Y_n$ and denote