Computers and Structures 154 (2015) 72–90

Contents lists available at ScienceDirect

Computers and Structures

journal homepage: www.elsevier.com/locate/compstruc

Computation of critical loads and buckling modes using hybrid-mixed stress finite element models

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ARTICLE INFO

Article history:
Received 3 April 2014
Accepted 15 February 2015
Available online 11 April 2015

Keywords:
Linear stability analysis
Hybrid-mixed formulation
Stress model
Legendre polynomials
Beams
Reissner–Mindlin plates

ABSTRACT

This work presents a hybrid mixed stress finite element model for the linear stability analysis of frame structures and Reissner–Mindlin plate bending problems. This model approximates the stress and displacement fields in the domain and the displacement field on the static boundary. Legendre polynomials are used to define all approximation bases, in order to compute analytical solutions for all structural operators. It enables also the implementation of highly efficient *p*-refinement procedures. A material linear behaviour is assumed and to validate the model and to illustrate its potential, numerical tests are presented and compared with analytical and numerical solutions.

1. Introduction

In recent years, hybrid-mixed stress finite element models have been developed for the static analysis of plate stretching and plate bending problems [9,11,21,22]. One of the main advantages associated with the use of this type of formulation is the flexibility introduced in the selection of the approximation functions. This fact makes possible the use of functions with special properties that cannot be implemented in the framework of conventional finite element models, such as Legendre polynomials [21,22] and systems of wavelets and Walsh series [8,11]. The hybrid-mixed stress models may lead also to *quasi-equilibrated* solutions, which are very convenient for design purposes. This type of alternative hybrid formulations has also been successfully applied over the years in the development of effective Trefftz models [15,16,19].

This paper presents a generalization of the hybrid-mixed stress model that allows for the geometric non-linear analysis of frame structures and Reissner–Mindlin plate bending problems. The model is called hybrid because both the stresses in the domain and the displacements on the static boundary (which includes the boundaries between elements) are simultaneously approximated. It is termed mixed because both the stresses and the displacements in the domain are directly approximated. All the fundamental conditions are enforced using a weighted residual approach designed to ensure that the discrete model embodies all the relevant properties presented by the continuum it represents, namely static-kinematic duality and elastic reciprocity.

Due to the type of enforcement followed for the equilibrium and compatibility conditions in the domain and because the connection between elements is ensured by the weighted residual enforcement of the equilibrium conditions on the boundary, the presented formulation is considered to lead to a stress model.

The model presented in this paper is based on the use of orthonormal Legendre polynomials as approximation functions. The properties of these functions allow for the definition of analytical closed form solutions for the computation of all structural operators. Numerical integration schemes are thus completely avoided. The numerical stability associated to the use of Legendre polynomial bases enables the use of meshes considering elements with large dimensions where the definition of highly effective *p*-adaptive refinement procedures is simplified. A detailed presentation of these functions can be found in [17,18].

The model reported in this paper makes possible the linearized stability analysis [25] for the computation of critical loads, buckling modes and geometrical nonlinear stresses and displacements. This paper starts with the presentation of the fundamental relations. Next, the hybrid-mixed stress model is discussed. Several strategies for the definition and computation of the nonlinear geometric operator are presented and discussed. The paper closes with the presentation of some numerical examples and the discussion of the main conclusions.

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http://dx.doi.org/10.1016/j.compstruc.2015.02.012
0045-7949/

2. Fundamental relations

Consider a set of forces acting in an elastic body. The equilibrium in the deformed position, the compatibility and the elasticity equations governing the behaviour of that structure may be expressed as follow [7,24]:

$$Ds + b + L_V = 0 \tag{2.1}$$

$$e = D^* u + L_E \tag{2.2}$$

$$e = fs \tag{2.3}$$

The vectors s, e and u list the independent components of the generalized stress, strain and displacement fields, respectively. The differential equilibrium operator, D, and the differential compatibility operator, D^*, represent linear and adjoint operators, and the vector b lists the body force components. L_V and L_E are general operators that take into account the geometric non-linear behaviour. In this paper, only the influence of operator L_V is taken into account.

For the frame and plate elements, the generalized stresses, strains, displacements and applied loads are defined by Eqs. (2.4) and (2.5).

$$s = \begin{Bmatrix} M \\ N \\ V \end{Bmatrix} e = \begin{Bmatrix} \chi \\ \varepsilon \\ \gamma \end{Bmatrix} u = \begin{Bmatrix} w \\ u \\ \theta \end{Bmatrix} b = \begin{Bmatrix} p \\ q \\ m \end{Bmatrix} \tag{2.4}$$

$$s = \begin{Bmatrix} m_x \\ m_y \\ m_{xy} \\ v_x \\ v_y \end{Bmatrix} e = \begin{Bmatrix} \chi_x \\ \chi_y \\ 2\chi_{xy} \\ \gamma_x \\ \gamma_y \end{Bmatrix} u = \begin{Bmatrix} \theta_x \\ \theta_y \\ w \end{Bmatrix} b = \begin{Bmatrix} m_1 \\ m_2 \\ q \end{Bmatrix} \tag{2.5}$$

The equilibrium, compatibility and flexibility operators are given by Eqs. (2.6) and (2.7) for the frame element and by Eqs. (2.8) and (2.9) for the case of Reissner–Mindlin plates.

$$D = \begin{bmatrix} 0 & 0 & \frac{d}{dx} \\ 0 & \frac{d}{dx} & 0 \\ \frac{d}{dx} & 0 & -1 \end{bmatrix} \quad D^* = \begin{bmatrix} 0 & 0 & \frac{d}{dx} \\ 0 & \frac{d}{dx} & 0 \\ \frac{d}{dx} & 0 & 1 \end{bmatrix} \tag{2.6}$$

$$f = \begin{bmatrix} \frac{1}{EI} & 0 & 0 \\ 0 & \frac{1}{EA} & 0 \\ 0 & 0 & \frac{1}{GA_C} \end{bmatrix} \tag{2.7}$$

$$D = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} & -1 & 0 \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & -1 \\ 0 & 0 & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \quad D^* = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 1 & 0 & \frac{\partial}{\partial x} \\ 0 & 1 & \frac{\partial}{\partial y} \end{bmatrix} \tag{2.8}$$

$$f = \frac{12}{Eh^3} \begin{bmatrix} 1 & -\nu & 0 & 0 & 0 \\ -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & \frac{(1+\nu)h^2}{6\phi} & 0 \\ 0 & 0 & 0 & 0 & \frac{(1+\nu)h^2}{6\phi} \end{bmatrix} \tag{2.9}$$

Fig. 1 represents the equilibrium in the deformed configuration for a single frame element. The geometric non-linear equilibrium condition in the domain is obtained by establishing the equilibrium conditions for an infinitesimal bar segment.

It is possible to obtain:

$$\begin{cases} \frac{dT}{dx} + p = 0 \\ \frac{dH}{dx} + q = 0 \\ \frac{dM}{dx} - T + H\frac{dw}{dx} + m = 0 \end{cases} \tag{2.10}$$

In order to express the equilibrium conditions in terms of the generalized stress resultants [29], it is necessary to take into account the rotation of the longitudinal axis, represented in Fig. 2 and defined according to [27].

This procedure ensures that the effect of shear stress resultant is considered in the computation of critical loads [27].

$$\begin{cases} \frac{dV}{dx} + \frac{d}{dx}\left(N\frac{dw}{dx}\right) + p = 0 \\ \frac{dN}{dx} - \frac{d}{dx}\left(V\frac{dw}{dx}\right) + q = 0 \\ \frac{dM}{dx} - (V + N\alpha) + (-V\alpha + N)\frac{dw}{dx} + m = 0 \end{cases} \Longleftrightarrow \begin{cases} \frac{dV}{dx} + \frac{d}{dx}\left(N\frac{dw}{dx}\right) + p = 0 \\ \frac{dN}{dx} - V\frac{d^2w}{dx^2} - \frac{dV}{dx}\frac{dw}{dx} + q = 0 \\ \frac{dM}{dx} - V - V\left(\frac{dw}{dx}\right)^2 + m = 0 \end{cases} \tag{2.11}$$

Assuming that the combined effect of shear force and quadratic small displacements are neglectable when computing buckling loads, it is possible to simplify the equilibrium equation to (2.12). As shear deformation is taken into account when defining the constitutive relations, it is still possible to compute the critical loads considering this effect [24].

$$\begin{cases} \frac{dV}{dx} + \frac{d}{dx}\left(N\frac{dw}{dx}\right) + p = 0 \\ \frac{dN}{dx} + q = 0 \\ \frac{dM}{dx} - V + m = 0 \end{cases} \tag{2.12}$$

For the modified (rotated) equilibrium position in the deformed shape, the geometric equilibrium operator defined in Eq. (2.1) can be written, for a frame element, as follows:

$$L_V = \begin{bmatrix} \frac{d}{dx}\left(N\frac{dw}{dx}\right) \\ 0 \\ 0 \end{bmatrix} \tag{2.13}$$

For a plate element, a similar procedure can be followed to obtain the domain equilibrium conditions in the deformed configuration. For this type of structural element, it is necessary to account for the influence of the in-plane stress in both directions and the effect of membrane shear. These effects are important as they may govern the critical loads and buckling modes, as explained in [28]. Fig. 3 shows all the stress fields that have to be involved in the definition of the equilibrium conditions, which be expressed as follows [29]:

$$\begin{cases} \frac{\partial m_x}{\partial x} + \frac{\partial m_{xy}}{\partial y} - t_x + h_x\frac{\partial w}{\partial x} + h_{xy}\frac{\partial w}{\partial y} + m_1 = 0 \\ \frac{\partial m_y}{\partial y} + \frac{\partial m_{xy}}{\partial x} - t_y + h_y\frac{\partial w}{\partial y} + h_{xy}\frac{\partial w}{\partial x} + m_2 = 0 \\ \frac{\partial t_x}{\partial x} + \frac{\partial t_y}{\partial y} + q = 0 \end{cases} \tag{2.14}$$

For the modified (rotated) equilibrium position in the deformed configuration, the geometric equilibrium operator in Eq. (2.1) can be written for the plate element as defined in Eq. (2.15), considering n_x, n_y and n_{xy} as constant in the domain [23].

$$L_V = \begin{bmatrix} 0 \\ 0 \\ n_x\frac{\partial^2 w}{\partial x^2} + 2n_{xy}\frac{\partial^2 w}{\partial x \partial y} + n_y\frac{\partial^2 w}{\partial y^2} \end{bmatrix} \tag{2.15}$$

The boundary of the structure may be subdivided into two complementary regions: (i) the kinematic boundary, Γ_u, in which the value for the displacement fields is prescribed and (ii) the static boundary, Γ_t, where the applied forces are prescribed. It can be written:

$$u = u_\Gamma \text{ on } \Gamma_u; \quad \mathcal{N}s + L_\Gamma = t \text{ on } \Gamma_t \tag{2.16}$$