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Unit root quantile autoregression testing using covariates

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ABSTRACT

This paper extends unit root tests based on quantile regression proposed by Koenker and Xiao [Koenker, R., Xiao, Z., 2004. Unit root quantile autoregression inference, Journal of the American Statistical Association 99, 775–787] to allow stationary covariates and a linear time trend. The limiting distribution of the test is a convex combination of Dickey–Fuller and standard normal distributions, with weight determined by the correlation between the equation error and the regression covariates. A simulation experiment is described, illustrating the finite sample performance of the unit root test for several types of distributions. The test based on quantile autoregression turns out to be especially advantageous when innovations are heavy-tailed. An application to the CPI-based real exchange rates using four different countries suggests that real exchange rates are not constant unit root processes.

1. Introduction

The unit root hypothesis has important implications for determining the effects of random shocks on economic variables. Recently, methods for detecting the presence of a unit root in semiparametric time series models have attracted interest in both theory and applications, since one way to increase power performance is the use of robust estimators, together with the associated inference apparatus. Such tests are designed to have a good power for many different error distributions. Thompson (2004), Koenker and Xiao (2004), Hasan (2001), Hasan and Koenker (1997), Rothenberg and Stock (1997), Herce (1996) and Lucas (1995) discuss robust estimation and testing in the presence of the unit root process.

Koenker and Xiao (2004) propose new tests of the unit root hypothesis based on the quantile autoregression (QAR) approach in an univariate context. Since many empirical applications have notoriously heavy-tailed behavior, it is important to consider estimation and inference procedures which are robust to departures from Gaussian conditions and are applicable to nonstationary time series. Quantile autoregression methods provide a framework for robust inference and allow one to explore a range of conditional quantiles exposing a variety of forms of conditional heterogeneity. Such models can still deliver important insights about dynamics and persistency in economic time series, and thus provide a useful tool in empirical diagnostic time series analysis. Koenker and Xiao (2004) suggest a t-ratio statistic to test the hypothesis of unit root that accounts only for intercept, and without covariates in the estimated equation. However, following Nelson and Plosser (1982), a common motivation for unit root testing is to test the hypothesis that a series is difference stationary against the alternative that it is trend stationary. Such tests are interesting because, under the alternative hypothesis of stationarity, time series exhibit trend reversion characteristics, whereas under the null they do not.

Hansen (1995) proposes a least squares based covariate augmented Dickey–Fuller (CADF) test, and shows that including correlated stationary covariates in the regression equation can lead to a more precise estimate of the autoregressive coefficient and consequently to large power gains. In this context, Elliott and Jansson (2003) and Pesavento (2007) propose generalizations of the CADF test. Therefore, another important extension of Koenker and Xiao (2004) is the inclusion of at least one covariate when testing for unit root.

This paper aims to generalize the quantile autoregression unit root test by introducing stationary covariates and a linear time trend into the quantile autoregression model. We explore estimation and inference in a model where there is one series that potentially has a unit root, and this series potentially covaries with some available stationary variable. The findings suggest that the limiting distribution of the t-ratio statistic based on quantile regression estimation after adding stationary covariates and a linear time trend continues to be a combination of Dickey–Fuller and Normal distributions, with weights determined by the correlation between the equation error and the regression covariates. Monte Carlo experiments show that the test based on covariate quantile autoregression (CQAR) turns out to be especially advantageous when innovations are non-Gaussian heavy-tailed. In particular, the results show that the quantile autoregression

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test proposed in this paper presents power gains relative to the QAR test proposed by Koenker and Xiao (2004) when there is an available stationary covariate and it is included in the estimated model. In addition, in the non-Gaussian heavy-tailed distribution case, the CQAR unit root test presents more power than the CADF test. Finally, we illustrate the test with an application to the CPI-based real exchange rates using four different countries: Canada, Japan, Switzerland and the United Kingdom. The results indicate that real exchange rates are not constant unit root processes.

The paper is organized as follows. In Section 2, we introduce the model and estimation. Section 3 presents the test and its asymptotic behavior, and in Section 4 we conduct a Monte Carlo experiment to study the performance of the estimator in finite sample. In Section 5 we apply the test to the CPI-based real exchange rates. Finally, Section 6 concludes the paper.

2. Quantile autoregression

2.1. The model and assumptions

The univariate series y_t consists of a deterministic and stochastic component

$$y_t = d_t + S_t, \tag{1}$$

for $t = 1, \dots, n$, where the deterministic component can be: $d_t = 0$, $d_t = \mu_1$, or $d_t = \mu_1 + \mu_2 t$. The stochastic component S_t is modeled as

$$a(L)\Delta S_t = \delta S_{t-1} + e_t, \tag{2}$$

where Δ is the usual difference operator, $a(L) = 1 - a_1 L - a_2 L^2 - \cdots - a_p L^p$ is a pth order polynomial in the lag operator, and

$$e_t = b(L)' x_t + u_t, \tag{3}$$

where x_t is a mean zero υ-vector, and $b(L) = b_{q_2} L^{-q_2} + \cdots + b_{q_1} L^{q_1}$ is a lag polynomial allowing for both leads and lags of x_t to enter the equation for e_t. So, we let the innovations (e_t) in the model be serially correlated, and also allow them to be related to other stationary covariates. We wish to test the unit root hypothesis $\delta = 0$ versus the alternative $\delta < 0$.

Assumptions: for some $p > r > 2$

A1. $\{u_t, x_t\}$ is covariance stationary and strong mixing with mixing coefficients α_m, which satisfies $\sum_{m=1}^{\infty} \alpha_m^{1/r-1/p} < \infty$;
A2. $\sup_t E[|x_t|^p + |u_t|^p] < \infty$;
A3. $E[x_{t-k} u_t] = 0$ for $q_1 \le k \le q_2$;
A4. $E[u_t u_{t-k}] = 0$ for $k \ge 1$;
A5. $E(\phi_t \phi_t') > 0$, where $\phi_t = (\Delta y_{t-1}, \dots, \Delta y_{t-q}, x_{t-q_2}, \dots, x_{t+q_1})'$;
A6. the roots of $a(L)$ all lie outside the unit circle;
A7. the distribution function of u_t, F, has differentiable continuous Lebesgue density, $0 < f(u) < \infty$, with bounded derivatives f' on $\{u : 0 < F(u) < 1\}$.

Assumptions A1–A6 are the same as in Hansen (1995). A1 and A2 state weak dependence and moment restrictions. Assumptions A3 and A4 exclude linear dependence. Assumption 6 is a typical stationarity assumption. Finally, assumption A7 is a standard assumption in quantile regression literature and imposes restriction on the density function of u_t.

2.2. Estimation

Estimation and testing are based on the following linear model[1]:

$$y_t = \mu_1 + \mu_2 t + \alpha y_{t-1} + \sum_{j=1}^{p} \alpha_j \Delta y_{t-j} + \sum_{l=-q_1}^{q_2} \gamma_l x_{t-l} + u_t. \tag{4}$$

The model may thus be written as

$$Q_{y_t}(\tau|\Im_{t-1}) = \mu_1 + \mu_2 t + \alpha y_{t-1} + \sum_{j=1}^{p} \alpha_j \Delta y_{t-j} + \sum_{l=-q_1}^{q_2} \gamma_l x_{t-l} + F_u^{-1}(\tau)$$

where Q_{y_t} denotes the τ-th conditional quantile of y_t conditional on $\Im_{t-1}$, where $\Im_{t-1}$ is the σ-field generated by $\{u_s, s < t, x_{t-q_2}, \dots, x_{t+q_1}\}$. The F_u denotes the common distribution function of the errors. Let $\mu_1(\tau) = \mu_1 + F_u^{-1}(\tau)$, and define

$$z_t = (1, t, y_{t-1}, \Delta y_{t-1}, \dots, \Delta y_{t-p}, x_{t-q_2}, \dots, x_{t+q_1})'$$
and $\beta(\tau) = (\mu_1(\tau), \mu_2, \alpha, \alpha_1, \dots, \alpha_p, \gamma_{q_1}, \dots, \gamma_{-q_2})'$,

thus, we have

$$Q_{y_t}(\tau|\Im_{t-1}) = z_t' \beta(\tau). \tag{5}$$

Estimation of the linear quantile autoregression model involves solving the problem

$$\min_{\beta \in R^{3+p+q}} \sum_{t=1}^{n} \rho_\tau \left(y_t - z_t' \beta \right), \tag{6}$$

where $\rho_\tau(u) = u(\tau - I(u < 0))$ as in Koenker and Bassett (1978). We shall be concerned with the limiting distribution of the coefficients in (6), more specifically with $\hat{\alpha}(\tau)$ and its t-ratio statistic under the hypothesis of unit root. Thus under the null hypothesis $\alpha(\tau) = 1$.

2.3. QAR asymptotics under unit root hypothesis

In this section we describe the limiting distribution of the quantile autoregression process under the unit root hypothesis, where the observations $\{y_t\}_{t=1}^{n}$ come from the data generating process as (1)–(3) with $d_t = \mu_1 + \mu_2 t$.[2] First, in order to derive the asymptotic properties of $\hat{\alpha}$, and without loss of generality, we use a convenient reparametrization of the objective function, by applying the quantile equivariance property. Further, we derive the asymptotic distribution of $\hat{\alpha}$.

Consider the estimator $\hat{\alpha}$ which solves

$$\min_{\beta \in R^{3+p+q}} \sum_{t=1}^{n} \rho_\tau \left(y_t - \mu_1 - \mu_2 t - \alpha y_{t-1} - \sum_{j=1}^{p} \alpha_j \Delta y_{t-j} - \sum_{l=-q_1}^{q_2} \gamma_l x_{t-l} \right). \tag{7}$$

Define $\tilde{y}_t = y_t - \mu_1 - \mu_2 t$. According to the equivariance property, Theorem 3.2 part 4 in Koenker and Bassett (1978), $\hat{\beta}(\tau, y, XA) = A^{-1} \tilde{\beta}(\tau, y, X)$, hence solving (7) is equivalent to

$$\min_{\beta \in R^{3+p+q}} \sum_{t=1}^{n} \rho_\tau \left(y_t - \eta - \theta t - \alpha \tilde{y}_{t-1} - \sum_{j=1}^{p} \alpha_j \Delta y_{t-j} - \sum_{l=-q_1}^{q_2} \gamma_l x_{t-l} \right), \tag{8}$$

where $\eta = \mu_1 + \alpha(\mu_2 - \mu_1)$, $\theta = \mu_2 + \alpha \mu_2$, and $\tilde{y}_{t-1} = y_{t-1} - \mu_1 - \mu_2(t-1)$. Therefore, $\hat{\alpha}$ which solves the minimization problem (7) also solves the minimization problem (8) and we can describe the asymptotic properties of $\hat{\alpha}$ based on the latter equation. Thus, let

$$\tilde{z}_t = (1, t, \tilde{y}_{t-1}, \Delta y_{t-1}, \dots, \Delta y_{t-p}, x_{t-q_2}, \dots, x_{t+q_1})' \tag{9}$$
and $\beta(\tau) = (\eta, \theta, \alpha, \alpha_1, \dots, \alpha_p, \gamma_{q_1}, \dots, \gamma_{-q_2})'$.

[1] When $d_t = \mu_1 + \mu_2 t$ the model (1)–(3) can be written as $a(L)\Delta y_t = \mu_1^* + \mu_2^* t + \delta y_{t-1} + b(L) x_t + u_t$, where $\mu_1^* = a(1)\mu_2 - \delta \mu_1$ and $\mu_2^* = -\delta \mu_2$. Since we have interest only in the autoregression coefficient, we omit the superscript.

[2] All results may be extended to the model generated by $d_t = \mu_1$ or $d_t = 0$.