Journal of Econometrics 152 (2009) 179–185

Contents lists available at ScienceDirect

Journal of Econometrics

journal homepage: www.elsevier.com/locate/jeconom

Quantiles, expectiles and splines

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ARTICLE INFO

Article history:
Available online 10 January 2009

JEL classification:
C14
C22

Keywords:
Asymmetric least squares
Cubic splines
Quantile regression
Signal extraction
State space smoother

ABSTRACT

A time-varying quantile can be fitted by formulating a time series model for the corresponding population quantile and iteratively applying a suitably modified state space signal extraction algorithm. It is shown that such quantiles satisfy the defining property of fixed quantiles in having the appropriate number of observations above and below. Like quantiles, time-varying expectiles can be estimated by a state space signal extraction algorithm and they satisfy properties that generalize the moment conditions associated with fixed expectiles. Because the state space form can handle irregularly spaced observations, the proposed algorithms can be adapted to provide a viable means of computing spline-based non-parametric quantile and expectile regressions.

1. Introduction

The movements in a time series may be described by time-varying quantiles. These may be estimated non-parametrically by fitting a simple moving average or a more elaborate kernel. An alternative approach is to formulate a partial model, the role of which is to focus attention on some particular feature – here a quantile – so as to provide a (usually nonlinear) weighting of the observations that will extract that feature by taking account of the dynamic properties of the series. The model is not intended to be taken as a full description of the distribution of the observations. Indeed models for different features, for example different quantiles, may not be consistent with each other.

In an earlier paper, we showed how time-varying quantiles could be fitted to a sequence of observations by setting up a state space model and iteratively applying a suitably modified signal extraction algorithm; see De Rossi and Harvey (2006). Here we determine the conditions under which such quantiles will satisfy the defining property of fixed quantiles in having the appropriate number of observations above and below.

Expectiles are similar to quantiles except that they are defined by tail expectations; see Newey and Powell (1987). Here we show how time-varying expectiles can be estimated by a state space signal extraction algorithm. This is similar to the algorithm used for quantiles, but estimation is more straightforward and much quicker. We then show that the conditions needed for a time-varying expectile to generalize the moment conditions associated with fixed expectiles are similar to those needed for a time-varying quantile to satisfy the defining property of fixed quantiles.

Section 2 reviews the ideas underlying fixed quantiles and expectiles. Section 3 then describes the signal extraction algorithms for estimating them when they are time-varying and establishes some basic properties. The final part of the paper is concerned with non-parametric estimation of regression models using splines. It has long been known that cubic splines can be fitted by signal extraction procedures because the state space form can handle irregularly spaced observations from a continuous time model. The proposed algorithms for time-varying quantiles and expectiles are easily adapted so as to provide a viable means of computing spline-based non-parametric quantile and expectile regressions. As well as illustrating the technique, we give a general proof of the equivalence between splines and the continuous time models underlying our signal extraction procedures for quantiles and expectiles.

2. Quantiles and expectiles

Let $\xi(\tau)$ – or, when there is no risk of confusion, ξ – denote the τth-quantile. The probability that an observation is less than $\xi(\tau)$ is τ, where $0 < \tau < 1$. Given a set of T observations, $y_t, t = 1, \dots, T$, (which may be from a cross-section or a time series), the sample quantile, $\widetilde{\xi}(\tau)$, can be obtained by sorting the observations in ascending order. However, it is also given as the solution to minimizing

$$S_\tau = \sum_{t=1}^{T} \rho_\tau(y_t - \xi) = \sum_{y_t < \xi} (\tau - 1)(y_t - \xi) + \sum_{y_t \geq \xi} \tau(y_t - \xi) \quad (1)$$

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doi:10.1016/j.jeconom.2009.01.001

with respect to ξ, where $\rho_\tau(.)$ is the *check function*, defined for quantiles as

$$\rho_\tau(y_t - \xi) = (\tau - I(y_t - \xi < 0))\,(y_t - \xi) \tag{2}$$

and $I(.)$ is the indicator function.

Expectiles, denoted $\mu(\omega)$, $0 < \omega < 1$, are similar to quantiles but they are determined by tail expectations rather than tail probabilities. For a given value of ω, the sample expectile, $\widetilde{\mu}(\omega)$, is obtained by minimizing the asymmetric least squares function,

$$S_\omega = \sum \rho_\omega(y_t - \mu) = \sum |\omega - I(y_t - \mu < 0)|\,(y_t - \mu)^2, \tag{3}$$

with respect to μ. Differentiating S_ω and dividing by minus two gives

$$\sum_{t=1}^{T} |\omega - I(y_t - \mu < 0)|\,(y_t - \mu). \tag{4}$$

The sample expectile, $\widetilde{\mu}(\omega)$, is the value of μ that makes (4) equal to zero. Setting $\omega = 0.5$ gives the mean, that is $\widetilde{\mu}(0.5) = \bar{y}$. For other ω's it is necessary to iterate.

3. Signal extraction

A framework for estimating time-varying quantiles, $\xi_t(\tau)$, can be set up by assuming that they are generated by stochastic processes and are connected to the observations through a measurement equation

$$y_t = \xi_t(\tau) + \varepsilon_t(\tau), \quad t = 1, \ldots, T, \tag{5}$$

where $\Pr(\varepsilon_t(\tau) < 0) = \tau$ with $0 < \tau < 1$. The disturbances, $\varepsilon_t(\tau)$, are assumed to be serially independent and independent of $\xi_t(\tau)$. The problem is then one of signal extraction. The assumption that the quantile or expectile follows a stochastic process can be regarded as a device for inducing local weighting of the observations. One possibility is a random walk,

$$\xi_t(\tau) = \xi_{t-1}(\tau) + \eta_t(\tau), \quad \eta_t(\tau) \backsim IID(0, \sigma^2_{\eta(\tau)}). \tag{6}$$

A smoother quantile can be extracted by a local linear trend

$$\begin{aligned} \xi_t &= \xi_{t-1} + \beta_{t-1} + \eta_t \\ \beta_t &= \beta_{t-1} + \zeta_t \end{aligned} \tag{7}$$

where β_t is the slope and ζ_t is $IID(0, \sigma^2_\zeta)$. It is well known that in a Gaussian model setting $\mathrm{Var}(\eta_t) = \sigma^2_\zeta/3$ and $\mathrm{Cov}(\eta_t, \zeta_t) = \sigma^2_\zeta/2$ results in the smoothed estimates being a cubic spline.

The model for expectiles is set up in a similar way with (5) replaced by $y_t = \mu_t(\omega) + \varepsilon_t(\omega)$ where the ω-expectile of $\varepsilon_t(\omega)$ is equal to zero.

3.1. Theory and computation

The state space form (SSF) for a univariate time series is:

$$\begin{aligned} y_t &= \mathbf{z}_t'\boldsymbol{\alpha}_t + \varepsilon_t, \qquad \mathrm{Var}(\varepsilon_t) = \sigma_t^2, \quad t = 1, \ldots, T \\ \boldsymbol{\alpha}_t &= \mathbf{T}_t\boldsymbol{\alpha}_{t-1} + \boldsymbol{\eta}_t, \qquad \mathrm{Var}(\boldsymbol{\eta}_t) = \mathbf{Q}_t \end{aligned} \tag{8}$$

where $\boldsymbol{\alpha}_t$ is an $m \times 1$ state vector, $\mathbf{z}_t$ is a non-stochastic $m \times 1$ vector, σ_t^2 is a non-negative scalar, $\mathbf{T}_t$ is an $m \times m$ non-stochastic transition matrix and $\mathbf{Q}_t$ is an $m \times m$ covariance matrix. The specification is completed by assuming that $\boldsymbol{\alpha}_1$ has mean $\mathbf{a}_{1|0}$ and covariance matrix $\mathbf{P}_{1|0}$ and that the serially independent disturbances ε_t and $\boldsymbol{\eta}_t$ are independent of each other and of the initial state.

Consider the criterion function

$$J = -\sum_{t=1}^{T} h_t^{-1}\rho(y_t - \mathbf{z}_t'\boldsymbol{\alpha}_t) - \frac{1}{2}\sum_{t=2}^{T} (\boldsymbol{\alpha}_t - \mathbf{T}_t\boldsymbol{\alpha}_{t-1})' \times \mathbf{Q}_t^{-1}(\boldsymbol{\alpha}_t - \mathbf{T}_t\boldsymbol{\alpha}_{t-1}) - \frac{1}{2}(\boldsymbol{\alpha}_1 - \mathbf{a}_{1|0})'\mathbf{P}_{1|0}^{-1}(\boldsymbol{\alpha}_1 - \mathbf{a}_{1|0}), \tag{9}$$

where $\rho(y_t - \mathbf{z}_t'\boldsymbol{\alpha}_t)$ is as in (2) or (3), with $\mathbf{z}_t'\boldsymbol{\alpha}_t$ equal to $\xi_t(\tau)$ or $\mu_t(\omega)$, $\mathbf{Q}_t$ and $\mathbf{P}_{1|0}$ are are assumed positive definite matrices as in (8) and h_t is a non-stochastic sequence of positive scalars. For example, in the local linear trend case (7) $\boldsymbol{\alpha}_t = (\xi_t, \beta_t)'$ and $\mathbf{z}' = (1, 0)$, while $\mathbf{T}$ is upper triangular with nonzero elements equal to one. Suppose that the initial state and the $\boldsymbol{\eta}_t's$ are normally distributed. For a Gaussian model of the form (8) the logarithm of the joint density of the observations and the states is, ignoring irrelevant terms, given by J with $\rho(y_t - \mathbf{z}_t'\boldsymbol{\alpha}_t) = (y_t - \mu_t(0.5))^2$ and $h_t = 2\sigma_t^2$. Differentiating J with respect to to each element of $\boldsymbol{\alpha}_t$ gives a set of equations, which, when set to zero and solved, gives the minimum mean square error estimates of $\boldsymbol{\alpha}_t$. These may be computed efficiently by the Kalman filter and associated smoother (KFS) as described in Durbin and Koopman (2001, pp. 70–73). If all the elements in the state are nonstationary and given a diffuse prior, the last term in J disappears. An algorithm is available as a subroutine in the SsfPack set of programs within Ox; see Koopman et al. (1999).

We can think of (9) as a criterion function that provides the basis for computing a quantile or expectile subject to a set of constraints imposed by the time series model for the quantile or expectile. [1] For expectiles differentiating J gives

$$\begin{aligned} \frac{\partial J}{\partial \boldsymbol{\alpha}_1} &= \mathbf{z}_1(2/h_1)IE(y_1 - \mathbf{z}_1'\boldsymbol{\alpha}_1) \\ &\quad - \mathbf{P}_{1|0}^{-1}(\boldsymbol{\alpha}_1 - \mathbf{a}_{1|0}) + \mathbf{T}_2'\mathbf{Q}_2^{-1}(\boldsymbol{\alpha}_2 - \mathbf{T}_2\boldsymbol{\alpha}_1) \\ \frac{\partial J}{\partial \boldsymbol{\alpha}_t} &= \mathbf{z}_t(2/h_t)IE(y_t - \mathbf{z}_t'\boldsymbol{\alpha}_t) - \mathbf{Q}_t^{-1}(\boldsymbol{\alpha}_t - \mathbf{T}_t\boldsymbol{\alpha}_{t-1}) \\ &\quad + \mathbf{T}_{t+1}'\mathbf{Q}_{t+1}^{-1}(\boldsymbol{\alpha}_{t+1} - \mathbf{T}_{t+1}\boldsymbol{\alpha}_t), \; t = 2, \ldots, T-1, \\ \frac{\partial J}{\partial \boldsymbol{\alpha}_T} &= \mathbf{z}_T(2/h_T)IE(y_T - \mathbf{z}_T'\boldsymbol{\alpha}_T) - \mathbf{Q}_T^{-1}(\boldsymbol{\alpha}_T - \mathbf{T}_T\boldsymbol{\alpha}_{T-1}) \end{aligned} \tag{10}$$

where

$$IE(y_t - \mu_t(\omega)) = |\omega - I(y_t - \mu_t(\omega) < 0)|\,(y_t - \mu_t(\omega)), \quad t = 1, \ldots, T. \tag{11}$$

The smoothed estimates, $\widetilde{\boldsymbol{\alpha}}_t$, satisfy the equations obtained by setting these derivatives equal to zero. Let $h_t = g_t/\kappa$, where κ is a constant, the interpretation of which will become apparent in sub-Section 3.3. For any expectile, adding and subtracting $\mathbf{z}_t g_t^{-1}\mathbf{z}_t'\boldsymbol{\alpha}_t$ to the equations in (10) allows the first term to be written as

$$\mathbf{z}_t g_t^{-1}[\mathbf{z}_t'\boldsymbol{\alpha}_t + 2\kappa IE\left(y_t - \mathbf{z}_t'\boldsymbol{\alpha}_t\right)] - \mathbf{z}_t g_t^{-1}\mathbf{z}_t'\boldsymbol{\alpha}_t, \quad t = 1, \ldots, T. \tag{12}$$

This suggests that we set up an iterative procedure in which the estimate of the state at the i-th iteration, $\widehat{\boldsymbol{\alpha}}_t^{(i)}$, is computed from the KFS applied to a set of synthetic 'observations' constructed as

$$\widehat{y}_t^{(i-1)} = \mathbf{z}_t'\widehat{\boldsymbol{\alpha}}_t^{(i-1)} + 2\kappa IE\left(y_t - \mathbf{z}_t'\widehat{\boldsymbol{\alpha}}_t^{(i-1)}\right). \tag{13}$$

The iterations are carried out until the $\widehat{\boldsymbol{\alpha}}_t^{(i)\prime}s$ converge whereupon $\widetilde{\mu}_t(\omega) = \mathbf{z}_t'\widetilde{\boldsymbol{\alpha}}_t$.

For quantiles, the first term in each of the three equations of (10) is replaced by $\mathbf{z}_t h_t^{-1} IQ(y_t - \mathbf{z}_t'\boldsymbol{\alpha}_t)$, where

$$IQ(y_t - \xi_t(\tau)) = \begin{cases} \tau - 1, & \text{if } y_t < \xi_t(\tau) \\ \tau, & \text{if } y_t > \xi_t(\tau) \end{cases} \quad t = 1, \ldots, T, \tag{14}$$

[1] It could also be regarded as the log of the joint density of a model where the measurement error is an asymmetric double exponential (quantile) or asymmetric normal (expectile). But such a model could not be taken seriously as a data generating process.