Journal of Econometrics 159 (2010) 209–221

Contents lists available at ScienceDirect

Journal of Econometrics

journal homepage: www.elsevier.com/locate/jeconom

Specification tests of parametric dynamic conditional quantiles☆

Juan Carlos Escanciano[a], Carlos Velasco[b,*]

[a] Economics Department, Indiana University, 100 S. Woodlawn, 47401, Bloomington, IN, USA

[b] Departamento de Economía, Universidad Carlos III de Madrid, Calle Madrid 126, 28903 Getafe (Madrid), Spain

ARTICLE INFO

Article history:
Received 5 August 2008
Received in revised form
8 June 2010
Accepted 20 June 2010
Available online 1 July 2010

JEL classification:
C12
C22

Keywords:
Omnibus tests
Conditional quantiles
Nonlinear time series
Empirical processes
Quantile processes
Subsampling
Value-at-risk
Tail risk

ABSTRACT

This article proposes omnibus specification tests of parametric dynamic quantile models. In contrast to the existing procedures, we allow for a flexible specification, where a possible continuum of quantiles is simultaneously specified under fairly weak conditions on the serial dependence in the underlying data-generating process. Since the null limit distribution of tests is not pivotal, we propose a subsampling approximation of the asymptotic critical values. A Monte Carlo study shows that the asymptotic results provide good approximations for small sample sizes. Finally, an application suggests that our methodology is a powerful alternative to standard backtesting procedures in evaluating market risk.

1. Introduction

Quantile regression is a powerful alternative to least squares regression in a wide range of econometric applications that vary from labor economics or demand analysis to finance; see the special issue of Empirical Economics (2001, vol. 26) and the references therein. Rather than relying on a single measure of conditional location, the quantile regression approach allows the researcher to explore a continuous range of conditional quantile functions, thereby providing a more complete and flexible analysis of the conditional dependence structure of the variables under consideration. A researcher interested in the whole conditional distribution will consider the specification of the conditional quantile at all quantile levels and some diagnostic on its global suitability. In fact, conditional goodness-of-fit tests are of paramount importance in econometrics and finance; see e.g. Andrews (1997) and Corradi and Swanson (2006). On the other hand, a risk manager will not be interested in the whole profit and loss account's distribution but mainly in its left tail, and hence will consider a set of small values of quantile levels, usually 1% or 5%, as recommended by the Basel Accord (1996a). Obviously, one can envision many situations in economics where the interest is in the lower or upper parts of the distribution; see, for instance, studies of unemployment duration (e.g. Koenker and Xiao (2002) and references therein), and wage inequalities (e.g. Machado and Mata, 2005). For these various situations, parametric quantile regressions have been shown to be a useful and flexible modelling strategy.

It is well known, however, that inference procedures within parametric quantile models depend crucially on the validity of the specified parametric functional forms for the range of quantiles under consideration. For instance, the counterfactual decomposition described in Machado and Mata (2005), which has been used recently in many studies to analyze the gender gap in log wages across the distribution (see e.g. Albrecht et al., 2007), and the martingale transform methods in Koenker and Xiao (2002) depend crucially on the linear quantile specification. Therefore, it is important to develop powerful tests for the correct specification of parametric conditional quantiles over a possibly continuous range of

☆ We would like to thank Roger Koenker and participants at the Econometrics in Rio Conference for helpful comments. Also, we would like to thank one referee, the associate editor and the joint editor for very helpful comments and suggestions. All errors are our own. Research funded by the Spanish Plan Nacional de I+D+I, reference number SEJ2007-62908.

* Corresponding author. Tel.: +34 91 6249646; fax: +34 91 6249875.
E-mail addresses: jescanci@iu.edu (J.C. Escanciano), carlos.velasco@uc3m.es (C. Velasco).

0304-4076/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.jeconom.2010.06.003

quantiles of interest and under fairly general conditions on the underlying data-generating process (DGP). This is the main purpose of the present paper.

More precisely, suppose that we observe a real-valued dependent variable Y_t, and the explanatory vector $I_{t-1} = (W'_{t-1}, Z'_t)' \in \mathbb{R}^d$, $d = s + m$, where $Z_t \in \mathbb{R}^m$, $m \in \mathbb{N}$, is an observable random vector (r.v.) and $W_{t-1} = (Y_{t-1}, \ldots, Y_{t-s})' \in \mathbb{R}^s$, where A' denotes the matrix transpose of A. We assume throughout the article that the time series process $\{(Y_t, Z'_t)' : t = 0, \pm 1, \pm 2, \ldots\}$, defined on the probability space $(\Omega, \mathcal{A}, P)$, is strictly stationary and ergodic. Assuming that the conditional distribution of Y_t given I_{t-1} is continuous, we define the α-th conditional quantile of Y_t given I_{t-1} as the measurable function q_α satisfying the conditional restriction

$$P(Y_t \le q_\alpha(I_{t-1}) \mid I_{t-1}) = \alpha, \quad \text{almost surely (a.s.)}, \alpha \in [0, 1]. \quad (1)$$

In *parametric* quantile regression modelling one assumes the existence of a family of functions $\mathcal{M} = \{m(\cdot, \theta(\alpha)) : \theta(\cdot) : \mathcal{T} \to \Theta \subset \mathbb{R}^p\}$, where $\mathcal{T}$ is a compact set which comprises the range of quantiles of interest, $\mathcal{T} \subset [0, 1]$, and one proceeds to make inference on θ or to test if $q_\cdot \in \mathcal{M}$, i.e., if there exists some $\theta_0 : \mathcal{T} \to \Theta$ such that $m(\cdot, \theta_0(\alpha)) = q_\alpha(\cdot)$ a.s. *for all* $\alpha \in \mathcal{T}$.[1]

Leading examples of specifications $\mathcal{M}$ are the linear quantile regression (LQR) model

$$m(I_{t-1}, \theta_0(\alpha)) \equiv m(Z_t, \theta_0(\alpha)) = Z'_t\theta_0(\alpha), \quad \alpha \in \mathcal{T},$$

with the *location-scale* regression model as the prominent example, in which $\theta_0(\alpha) = (\beta_0, \gamma_0 F_0^{-1}(\alpha)) \in \Theta \subset \mathbb{R}^p, m = p$, and where $F_0^{-1}(\alpha)$ denotes a univariate quantile function, see, e.g., Koenker and Xiao (2002), or the linear quantile autoregression model of order s (LQAR(s)), where

$$m(I_{t-1}, \theta_0(\alpha)) \equiv m(W_{t-1}, \theta_0(\alpha)) = \theta_{01}(\alpha) + W'_{t-1}\theta_{02}(\alpha),$$
$$\theta_0(\alpha) = (\theta_{01}(\alpha), \theta'_{02}(\alpha))',$$

which results, for instance, from the random coefficient model

$$Y_t = \theta_{01}(U_t) + W'_{t-1}\theta_{02}(U_t), \quad (2)$$

where $\theta_{01}(\cdot)$ and $\theta_{02}(\cdot)$ are such that the right-hand side of (2) is monotone increasing in U_t, and $\{U_t\}$ are independent and identically distributed (i.i.d.) $U[0, 1]$ random variables; see Koenker and Xiao (2006) for inferences on the LQAR(s) model.

Much effort has been devoted to inferences on $\theta_0(\alpha)$ for the aforementioned models based on the associated quantile processes $Q_n(\alpha) := \sqrt{n}\,(\theta_n(\alpha) - \theta_0(\alpha))$, for $\theta_n(\alpha)$ a $\sqrt{n}$-consistent estimator of $\theta_0(\alpha)$. It is well known, however, that inferences based on $Q_n(\alpha)$ will heavily depend on the correct specification of the parametric quantile regression model. Although there exist some works on quantile regression model checks, to the best of our knowledge no consistent test for $q_\cdot \in \mathcal{M}$ has been proposed. The existing literature has been mostly limited to i.i.d. observations, linear models, and more importantly to a fixed quantile level $\alpha \equiv \alpha_0 \in (0, 1)$. In particular, Zheng (1998) has proposed a quantile regression specification test based on kernel smoothing estimators of the conditional moment $E[1(Y_t \le m(I_{t-1}, \theta_0(\alpha_0))) - \alpha_0 \mid I_{t-1}]$; see also Horowitz and Spokoiny (2002) for the median function (i.e., $\alpha_0 = 0.5$). Recently, Whang (2005), using empirical likelihood methods, has proposed a specification test for quantile regression and censored quantile regression for i.i.d. data. Tests based on smoothers usually have known asymptotic null distributions after an appropriate choice of the bandwidth sequence, but they are not consistent against Pitman's local alternatives.

Using an integrated approach, Bierens and Ginther (2001) proposed a diagnostic test for a linear quantile regression. These authors consider i.i.d. observations and do not take into account the uncertainty due to parameter estimation. Their test is consistent against $n^{-1/2}$ local alternatives, with n the sample size, but it relies on an upper bound on the asymptotic critical value, which might be too conservative. To solve this deficiency, Whang (2006) considered a subsampling approach to approximate the asymptotic critical values. Koul and Stute (1999) introduced asymptotic pivotal tests for parametric conditional quantiles of first-order nonlinear autoregressive processes. To obtain the pivotal property of the test they use a martingale transform (cf. Khmaladze, 1981). Alternatively, He and Zhu (2003) developed a bootstrap-based test for linear and nonlinear quantile regressions. Our paper also contributes to this literature of specification tests for a unique quantile, since our methods trivially apply to the unique quantile case in a more general framework than the previously cited works. By extending the scope of conditional quantile specifications to a possible continuum of quantiles, we provide a new and flexible specification procedure.[2]

In the present article, we propose omnibus tests for $q_\cdot \in \mathcal{M}$ that are valid for general linear and nonlinear quantile models under time series. Our tests are based on the fact that $q_\cdot \in \mathcal{M}$ is characterized by the *infinite* set of conditional moment restrictions

$$E[1(Y_t \le m(I_{t-1}, \theta_0(\alpha))) - \alpha \mid I_{t-1}]$$
$$= 0 \quad \text{a.s. for some } \theta_0(\cdot) : \mathcal{T} \to \Theta \subset \mathbb{R}^p,\ \forall \alpha \in \mathcal{T}. \quad (3)$$

The proposed tests are functionals of a quantile-marked empirical process that characterizes condition (3). The asymptotic theory is largely complicated by the fact that (3) involves an infinite number of conditional moment restrictions, indexed by $\alpha \in \mathcal{T}$. We solve this technical difficulty using delicate weak convergence results for empirical processes under martingale conditions. It turns out that the asymptotic null distributions of test statistics depend on the specification under the null and the DGP. Therefore, we propose to implement the test with the assistance of the subsampling.

The rest of the article is organized as follows. In Section 2, we introduce the quantile-marked process, which is the basis upon which the new test statistics for testing (3) are developed. We study the asymptotic distribution of the proposed tests under the null, fixed and local alternatives. In Section 3, a subsampling procedure for approximating the asymptotic null distribution of tests is considered. In Section 4, we present a simulation exercise assessing the finite-sample performance of tests. Section 5 summarizes the results of an application to some European stock indexes, showing that our methodology can serve as a powerful and flexible alternative to standard backtesting procedures in evaluating market risk. Finally, Section 6 concludes. Proofs are deferred to an Appendix. Throughout the article, A^c and $|A|$ denote the complex conjugate and Euclidean norm of A, respectively. In what follows, C is a generic constant that may change from one expression to another. All limits are taken as $n \to \infty$.

1 We can actually take $\mathcal{T} = [0, 1]$ in our theory provided that the centered estimator $\sqrt{n}(\theta_n - \theta_0)$ is asymptotically tight on the whole interval [0, 1]. To the best of our knowledge, such a result is, however, not available in the literature for most popular estimators. Thus, we do not pursue such generality in this paper and we restrict our analysis to $\mathcal{T} \subset [0, 1]$, in accordance with the econometrics literature.

2 During the revision of this paper, one referee pointed out an unpublished Ph.D. dissertation by Nejmeldeen (2003) proposing specification tests for a continuum of quantiles in a linear quantile model. Nejmeldeen's (2003) results require high-level conditions. In particular, his Assumption (E) directly assumes that a certain empirical process is stochastic equicontinuous, which is certainly the most difficult part and the bulk in the development of our specification tests. In contrast, we prove the stochastic equicontinuity condition under a set of primitive regularity assumptions and for a more general setting than that considered in Nejmeldeen (2003).