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# Likelihood inference for a nonstationary fractional autoregressive model

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ABSTRACT

This paper discusses model-based inference in an autoregressive model for fractional processes which allows the process to be fractional of order $d$ or $d-b$. Fractional differencing involves infinitely many past values and because we are interested in nonstationary processes we model the data $X_1, \ldots, X_T$ given the initial values $X_{-n},\ n = 0, 1, \ldots$, as is usually done. The initial values are not modeled but assumed to be bounded. This represents a considerable generalization relative to previous work where it is assumed that initial values are zero. For the statistical analysis we assume the conditional Gaussian likelihood and for the probability analysis we also condition on initial values but assume that the errors in the autoregressive model are i.i.d. with suitable moment conditions.

We analyze the conditional likelihood and its derivatives as stochastic processes in the parameters, including $d$ and $b$, and prove that they converge in distribution. We use these results to prove consistency of the maximum likelihood estimator for $d, b$ in a large compact subset of $\{1/2 < b < d < \infty\}$, and to find the asymptotic distribution of the estimators and the likelihood ratio test of the associated fractional unit root hypothesis. The limit distributions contain the fractional Brownian motion of type II.


## 1. Introduction and motivation

Nonstationary autoregressive models have been studied intensively over the past three decades. In the usual autoregressive model, we consider the nonstationarity implied by a unit root in the associated autoregressive polynomial. Recently, much attention has been given to alternative models of nonstationarity such as fractional models characterized by a fractional rather than an integer integration operator, see e.g. Baillie (1996) or Henry and Zaffaroni (2003) for reviews and examples, and it appears important to allow fractional orders of integration (or fractionality) in the time series model. Formal statistical tests of the unit root hypothesis are of additional interest to economists because they can help to evaluate the nature of the nonstationarity that most macroeconomic and financial time series data exhibit.

The autoregressive model with $k+1$ lags for the univariate time series $X_t,\ t = 1, \ldots, T$, conditional on initial values $X_{-n},\ n = 0, \ldots, k$, written in levels and differences, is

$$\Delta X_t = \pi X_{t-1} + \sum_{i=1}^{k} \phi_i \Delta X_{t-i} + \varepsilon_t, \quad t = 1, \ldots, T,$$

where $\varepsilon_t$ is i.i.d. $(0, \sigma^2)$.

A corresponding *fractional autoregressive model*, conditional on infinitely many initial values $X_{-n}, n = 0, 1, \ldots$, was obtained by Johansen (2008) by replacing the difference and lag operators $\Delta$ and $L = 1 - \Delta$ by their fractional counterparts $\Delta^b$ and $L_b = 1 - \Delta^b$ and applying $\Delta^{d-b}$ to $X_t$:

$$\Delta^d X_t = \pi \Delta^{d-b} L_b X_t + \sum_{i=1}^{k} \phi_i \Delta^d L_b^i X_t + \varepsilon_t, \quad t = 1, \ldots, T. \tag{1}$$

In the statistical model defined by (1) the parameters $(d, b, \phi_1, \ldots, \phi_k, \pi, \sigma^2)$ are unrestricted except $d \geq b > 1/2$ and $\sigma^2 > 0$. Note that if $k = 0$ in model (1) the parameter $b$ is not identified under the unit root null $\pi = 0$, see Section 2.3. This motivates the study of the simpler model with $d = b$,

$$\Delta^d X_t = \pi L_d X_t + \sum_{i=1}^{k} \phi_i \Delta^d L_d^i X_t + \varepsilon_t, \quad t = 1, \ldots, T, \tag{2}$$

and in the simplest case with $k = 0$, model (2) reduces to

$$\Delta^d X_t = \pi L_d X_t + \varepsilon_t, \quad t = 1, \ldots, T, \tag{3}$$

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which we consider separately in some of our results.

The properties of the solution of model (1) depend on the characteristic function for the process,

$$\begin{aligned}\Pi(z) &= (1-z)^d - \pi(1-z)^{d-b}(1-(1-z)^b) \\ &\quad - \sum_{i=1}^{k}\phi_i(1-z)^d(1-(1-z)^b)^i \\ &= (1-z)^{d-b}\xi(1-(1-z)^b) = \sum_{j=-1}^{k}\rho_j(1-z)^{d+jb}, \end{aligned} \quad (4)$$

$$\xi(y) = 1 - y - \pi y - \sum_{i=1}^{k}\phi_i(1-y)y^i = \sum_{j=0}^{k+1}\rho_{j-1}(1-z)^{jb},$$

where $\sum_{j=-1}^{k}\rho_j = 1$ and $y = 1-(1-z)^b$. Note that $\Pi(z)$ is a polynomial in $z$ if and only if $d$ and $b$ are non-negative integers, whereas $\xi(y)$ is a polynomial in $y$ for any $d, b$. The parameter $\rho = (\rho_{-1}, \ldots, \rho_k)'$ is a simple linear transformation of $\pi$ and $\phi = (\phi_1, \ldots, \phi_k)'$ and satisfies $\rho_{-1} = -\pi$, $\rho_k = (-1)^{k+1}\phi_k$ and $\sum_{j=-1}^{k}\rho_j = 1$.

As a statistical model we analyze the conditional distribution of $(X_1, \ldots, X_T)$ given the initial values $X_{-n}$, $n = 0, 1, \ldots$, under the assumption that $\varepsilon_t$ is i.i.d. $N(0, \sigma^2)$. That is, we derive estimators and test statistics from the conditional likelihood function and their properties from the conditional distribution. For the asymptotic analysis we assume that $\varepsilon_t$ is i.i.d. with suitable moment conditions and that $X_{-n}$ is bounded. Thus, the initial values are not modeled but treated as (bounded) constants. The standard approach in the literature conducts inference conditional on initial values, which are set equal to zero. In this paper we develop methods for analysis of the nonstationary (unit root) case with $\pi = 0$ and $d \geq b > 1/2$. We call the test of $\pi = 0$ the fractional unit root test in our model.

Our main result is to find asymptotic properties of (Gaussian) maximum likelihood estimators of the parameters in model (1) under the assumption that $\pi = 0$, and the asymptotic distribution of the likelihood ratio (LR) test that $\pi = 0$. We show that if the initial values are bounded they have no influence on limit results, except that conditioning on initial values implies that some of the limit results are expressed in terms of the fractional Brownian motion (fBM) of type II, whereas fBM of type I plays no role in the analysis.

Thus, (1) and (2) are fractional versions of the augmented Dickey and Fuller (1979, 1981) regression model, and we provide a first analysis of a univariate model with two fractional parameters with a simple criterion for different orders of fractionality. The test that $\pi = 0$ is a test that the process is fractional of order $d$ versus $d - b$, i.e. the fractional unit root test is also a test of the order of fractionality of $X_t$. Note that when $d > b$ the characteristic function (4) has a unit root also when $\pi \neq 0$. However, we still refer to the test of $\pi = 0$ as the unit root test in (1) since it is a test of a unit root in the polynomial $\xi(y)$. Other hypotheses of interest are linear hypotheses on the regression parameters $\phi = (\phi_1, \ldots, \phi_k)'$ and the fractionality parameters $d$ and $b$. The analysis of model (1) is a first step towards the analysis of the multivariate model which allows cofractionality, that is, linear combinations of fractional processes that decrease the order of fractionality.

A prominent place in the literature on models for fractional processes is held by the ARFIMA model

$$A(L)\Delta^d X_t = B(L)\varepsilon_t, \quad (5)$$

where $A(L)$ and $B(L)$ are the autoregressive and moving average polynomials, respectively. The ARFIMA model generalizes the well-known ARIMA model by introducing the fractional (non-integer) order of differencing, $d$. The original Dickey and Fuller (1979, 1981) test is an LR test of $A(1) = 0$ within the autoregressive model with $d = 0$ and $B(L) = 1$. A Wald-type test of the same null was considered by, e.g., Chan and Terrin (1995) and Ling and Li (2001) in the ARFIMA model, where the null hypothesis $A(1) = 0$ implies that the process is fractional of order $d + 1$ versus order $d$ under the alternative.[2]

The model we propose to analyze, (1), is different from the ARFIMA model (5) because it is characterized by two fractional parameters and because of the role of the lag operator $L_b$. Our model is not an ARFIMA model in $L$, unless $b = 0, 1, 2, \ldots$, but an ARFIMA model in the new lag operator $L_b$. Thus, in (5) there is only one fractional parameter and the fractional order of the time series always differs by exactly one under the unit root null and the alternative, whereas the lag operator $L_b$ implies that the difference in order of fractionality of the process generated by model (1) under the unit root null and the alternative is $b$ rather than one, see Section 2.1 below. Note that when the roots of $A(z)$ in (5) are outside the unit circle, we find that $X_t$ is fractional of order $d$. Thus, a simple criterion exists for fractionality of the solution of the ARFIMA model. A similar simple condition exists in terms of $\xi(y)$ for the process generated by model (1), see Lemma 1.

Another strand of the literature analyzes regression-type statistics with the purpose of testing for a fractional unit root. An early contribution is Sowell (1990) who analyzed the regression $y_t = \phi y_{t-1} + u_t$, where $\Delta^d u_t = \varepsilon_t$ and $\phi_0 = 1$. He derived the asymptotic distribution of $\hat{\phi}_{FS}$, the regression estimator of $y_t$ on $y_{t-1}$, instead of the maximum likelihood estimator for fixed $d$, $\hat{\phi}_{ML}$, that is, a regression of $\Delta^d y_t$ on $\Delta^d y_{t-1}$ as considered by Ling and Li (2001). Consequently, the asymptotic distribution of the estimator $\hat{\phi}_{FS}$ is discontinuous in $d$, in the sense that $T^{2d+1}(\hat{\phi}_{FS} - 1)$ converges in distribution to a fBM functional when $d \leq 0$ and $T(\hat{\phi}_{FS} - 1)$ converges in distribution to another such functional when $d \geq 0$. On the other hand, the distribution of $\hat{\phi}_{ML}$ is the same as that of the standard Dickey and Fuller (1979, 1981) statistic (see also the analysis in Phillips, 1987). The ideas in Sowell (1990) were further developed by Dolado et al. (2002) who consider the statistical model $\alpha(L)\Delta y_t = \phi\Delta^{d_1}y_{t-1} + \varepsilon_t$ and test that $\phi = 0$, and Lobato and Velasco (2006) who consider the model $\alpha(L)\Delta y_t = \phi\alpha(L)(\Delta^{d_2-1} - 1)\Delta y_t + u_t$ and test that $\phi = 0$. Here $\alpha(L)$ is a lag polynomial. They indicate the properties of the process under the null and under the alternative.[3] In both cases they apply a $t$-ratio based on a regression equation and motivated by the model equations, rather than a test based upon an analysis of the likelihood function.

Model (1) proposed here has the advantage relative to that of Dolado et al. (2002) and others, that one can give simple criteria for fractional integration of various orders in terms of the parameters of the model, see Johansen (2008, Theorem 8) and Lemma 1. In this way we have a platform for conducting model-based statistical inference on the parameters and on the fractional order of $X_t$, and the possibility of extending the results to the multivariate case.

The remainder of the paper is organized as follows. In Section 2 we discuss the properties of the solution of model (1), including

[2] Robinson (1994) proposed testing for a unit root using the LM test in several different models, see also Tanaka (1999) and Nielsen (2004). However, these authors examined the properties of hypothesis tests of the form $d = d_0$ (against composite alternatives) in ARFIMA models, so these are not unit root tests in the sense defined above.

[3] The condition given by Dolado et al. (2002) for the roots of $\pi(z) = \alpha(z)(1-z)^{1-d_1} - \phi z = 0$ to be outside the unit circle is $\pi(0) = 1, \pi(1) > 0, \pi(-1) > 0$. This cannot be correct as the example $\pi(z) = 4(z - 1/2)^2 = (1-4z)(1-z) + z$ shows. Indeed, the correct condition leads to the solution of an unpleasant transcendental equation, see the discussion in Johansen (2008), and thus, it does not appear possible to give simple conditions for fractionality of various orders in terms of the parameters of their model.