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A recovery-type *a posteriori* error estimator for gradient elasticity

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ABSTRACT

In this paper, an *a posteriori* error estimator of the recovery type is developed for the gradient elasticity theory of Aifantis. This version of gradient elasticity can be implemented in a staggered way, whereby solution of the classical equations of elasticity is followed by solving a reaction–diffusion equation that introduces the gradient enrichment and removes the singularities. With gradient elasticity, singularities in the stress field can be avoided, which simplifies error estimation. Thus, we develop an error estimator associated with the second step of the staggered algorithm. Stress-gradients are recovered based on the methodology of Zienkiewicz and Zhu, after which a suitable energy norm is discussed. The approach is illustrated with a number of examples that demonstrate its effectiveness.

1. Introduction

In classical elasticity, the stresses depend only on the first order derivative of displacements (strains) and not on higher-order derivatives. No information on the material's microstructure is present in classical elasticity, and as a consequence size-dependent behaviour cannot be captured with classical elasticity. Moreover, classical elasticity is plagued by the occurrence of singular stresses and strains at the tips of sharp cracks, re-entrant corners or where point loads are applied. An alternative to classical elasticity is so-called *gradient elasticity*, in which the field equations are equipped with additional higher-order spatial derivatives of the relevant state variables. The higher-order terms are accompanied by an additional material parameter with the dimensions of length – this parameter is linked to the micro-structural geometry and is called "internal length scale". Due to the presence of such an internal length scale, size-dependent mechanical behaviour can be described [5,7]. Furthermore, the occurrence of singularities in the stress and strain field can be avoided with gradient elasticity.

One of the most versatile variants of gradient elasticity theory is the Aifantis theory [1–3]. Its attractiveness is due to its mathematical structure, which allows the fourth-order equilibrium equations to be solved as an uncoupled sequence of two sets of second-order equations [3]. For numerical implementations, this has the significant consequences that simple, C^0-continuous interpolations suffice for the spatial discretisation. This makes finite element implementation straightforward, as has been demonstrated in a number of studies [4–8].

We can use *a priori* and *a posteriori* error estimation techniques in order to determine the accuracy of numerical solutions. A systematic comparative presentation of these techniques is given in [9], and an in-depth discussion on various types of estimators can be obtained in [18]. The two main families of *a posteriori* error estimators are the residual type estimators [10,11] and the recovery type estimators [12–14] – here, the discussion will focus on the latter. The recovery type error estimators have been first introduced by Zienkiewicz and Zhu [12] and later, the authors presented the so-called superconvergent patch recovery method which improved the performance of recovery based methods [13,14]. This error estimate can also be applied to hierarchical *p*-refinement with a slight modification as given in [15]. A local *a posteriori* error estimator for the extended finite element method is devised in [16] which is based on a derivative recovery technique in the L_2 norm and is applied to linear elastic fracture mechanics. In their later study, the authors proposed an extended global derivative recovery technique for extended finite elements [17].

Thus, recovery type error estimators can be devised for use in fracture mechanics, where singularities are known to exist in the solution. However, and as argued above, it is also possible to analyse cracks with gradient elasticity by which singularities in the stress field can be avoided altogether. This should facilitate error estimation, and in this study a recovery type *a posteriori* error estimator for gradient elasticity will be developed. After revisiting the basic equations of the Aifantis gradient elasticity theory in Section 2 and its finite element implementation in Section 3, the suggested error estimator will be discussed in Section 4. The

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effectiveness of this approach is demonstrated with two benchmark problems in Section 5.

2. Aifantis' gradient elasticity theory

One of the most popular gradient elasticity theories is the one derived by Aifantis and co-workers in the early 1990s [1–3]. In this theory, the usual linear elastic constitutive relations are extended with the Laplacian of the strain as

$$\sigma_{ij} = C_{ijkl}\left(\varepsilon_{kl} - \ell^2 \varepsilon_{kl,mm}\right) \tag{1}$$

where σ is the Cauchy stress, C is the constitutive tensor, ε is the usual infinitesimal strain and ℓ is an internal length scale parameter representing the microstructure of the material. The equilibrium equations can be written in terms of displacement derivatives as

$$C_{ijkl}\left(u_{k,jl} - \ell^2 u_{k,jlmm}\right) + b_i = 0 \tag{2}$$

where b are the body forces. The attractiveness of this theory is (i) that it contains only one internal length scale parameter, and (ii) that its mathematical structure allows to solve the fourth-order partial differential equations as an uncoupled sequence of two sets of second-order equations. More specifically, the various derivatives in Eq. (2) can be factorised, so that Eq. (2) can be rewritten as

$$C_{ijkl}u^c_{k,jl} + b_i = 0 \tag{3}$$

which are the equations of classical elasticity, followed by

$$u^g_k - \ell^2 u^g_{k,mm} = u^c_k \tag{4}$$

Here u^c are the displacements following from the classical elasticity equations, whilst u^g are the gradient-enriched displacements. Note that u^g in Eq. (4) is identical to u in Eq. (2), and the superscript g is used to distinguish the gradient-enriched displacements from its classical counterpart.

If Eq. (4) is substituted back into Eq. (3), it is easily verified that Eq. (2) is retrieved. When this operator split was suggested first by Ru and Aifantis [3], the gradient enrichment was expressed in terms of displacements as given in Eq. (4). However, it can also be evaluated in terms of stresses by differentiation as

$$\sigma^g_{ij} - \ell^2 \sigma^g_{ij,mm} = C_{ijkl}u^c_{k,l} \tag{5}$$

The use of Eq. (5) instead of Eq. (4) has some advantages: it was demonstrated in [5,6] that the use of Eq. (4) does not necessarily remove the singularities from all stress components at the tip of sharp cracks, whereas all stress singularities are removed if Eq. (5) is used – this discrepancy can be attributed to the nature of the variationally consistent boundary conditions [5].

3. Implementation of Aifantis' theory

In this section, matrix–vector notation will be used instead of index notation, as is customary in finite element literature. A finite element implementation of the Aifantis' theory which is based on Eq. (4) was first given in [4] and then extended to include Eq. (5) in [5]. In this study, and following the recommendations in [5], Eqs. (3) and (5) are used for the implementation of the Aifantis theory of gradient elasticity.

Eq. (3) is the usual expression of equilibrium in classical elasticity, the spatial discretisation of which is well known and does not need to be repeated here. The weak form of Eq. (5) is obtained by premultiplying with a virtual strain field δe and integrating over the domain Ω as

$$\int_\Omega \delta e^T \cdot \left(\sigma - \ell^2 \nabla^2 \sigma - CLu\right) dV = 0 \tag{6}$$

where L is the usual strain–displacement differential operator, which in the two dimensional case is defined as

$$L^T = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \tag{7}$$

Integrating by parts and substituting $\delta e = S\delta\sigma$, where $S = C^{-1}$, results in

$$\int_\Omega \delta\sigma^T S\sigma dV + \int_\Omega \left(\frac{\partial \sigma^T}{\partial x} Sl^2 \frac{\partial \sigma}{\partial x} + \frac{\partial \sigma^T}{\partial y} Sl^2 \frac{\partial \sigma}{\partial y}\right) dV - \int_\Omega \delta\sigma^T L u dV = 0 \tag{8}$$

Here, the boundary terms are ignored, which is equivalent to adopting the homogeneous natural boundary condition $n \cdot \nabla\sigma = 0$ [5,7]. Finite element discretisation of Eq. (8) gives

$$\underline{\delta\sigma}^T \int_\Omega \left(N^T_\sigma S N_\sigma + \frac{\partial N^T_\sigma}{\partial x} S\ell^2 \frac{\partial N_\sigma}{\partial x} + \frac{\partial N^T_\sigma}{\partial y} S\ell^2 \frac{\partial N_\sigma}{\partial y}\right) dV \underline{\sigma} = \underline{\delta\sigma}^T \int_\Omega N^T_\sigma B_u dV \underline{u} \tag{9}$$

The two fields of unknowns, namely the classical displacements and the gradient-enriched stresses, are discretised with shape functions N_u and N_σ, respectively. Furthermore, $B_u = LN_u$ and underlined vectors contain the discretised nodal values of their continuous counterparts. With these specifications, the resulting system of equations can be written as

$$\begin{bmatrix} K_{uu} & 0 \\ -K^T_{u\sigma} & K_{\sigma\sigma} \end{bmatrix} \begin{bmatrix} \underline{u} \\ \underline{\sigma} \end{bmatrix} = \begin{bmatrix} \underline{f} \\ \underline{0} \end{bmatrix} \tag{10}$$

where $\underline{f}$ is the external force vector, and

$$K_{uu} = \int B^T_u C B_u dV \tag{11}$$

$$K_{u\sigma} = \int B^T_u N_\sigma dV \tag{12}$$

$$K_{\sigma\sigma} = \int \left(N^T_\sigma S N_\sigma + \frac{\partial N^T_\sigma}{\partial x} S\ell^2 \frac{\partial N_\sigma}{\partial x} + \frac{\partial N^T_\sigma}{\partial y} S\ell^2 \frac{\partial N_\sigma}{\partial y}\right) dV \tag{13}$$

Eq. (10) is a decoupled system of equations in which the first row of equations can be solved prior to the second row of equations. Thus, it can be said that the gradient-enrichment (second row of equations) constitutes a post-processing of the results of classical elasticity (first row of equations).

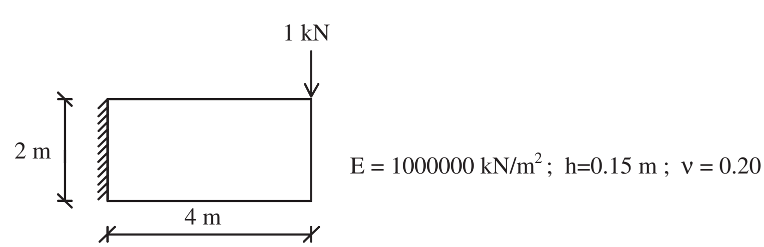

Fig. 1. Cantilever beam under a tip load of 1 kN.