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Axiomatic properties of geo-logarithmic price indices

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ABSTRACT

In the axiomatic approach to composite index numbers, a list of properties is given that both price and quantity indices should satisfy in order to ensure consistent comparisons. Usually, the price index is selected first and its cofactor is consequently adopted as the (implicit) quantity index. Unfortunately, even if the price index has good axiomatic properties, its cofactor need not, so the implicit quantity comparison may be axiomatically inconsistent. In this paper, we give a comprehensive study of a family of price indices sharing good axiomatic properties (proportionality, commensurability, and homogeneity) together with their cofactors. This family, called geo-logarithmic, is relevant also because of the empirical circumstance that all known price indices sharing such properties with their cofactors belong to it or can be obtained from geo-logarithmic index numbers through simple transformations. Thus, the geo-logarithmic family seems to play a central role when the joint consistency of price and quantity comparisons is concerned.

1. Introduction

In the axiomatic approach to composite index numbers (Balk, 1995), price and quantity indices are dually linked since their product has to decompose the value index in a multiplicative way. In practice, the price index is often given some prominence, so that its formula is selected first and its cofactor is *de facto* chosen as the quantity index, ensuring the value index decomposition to hold. This fact introduces a small "symmetry breaking" in the duality of price and quantity indices, which is reasonable in itself, since often price comparison is the main interest. Unfortunately such an induced asymmetry can also have unexpected consequences. Usually no requirement is given connecting the choice of the price index to the properties actually satisfied by its cofactor, so the former is very often selected irrespective of the axiomatic properties fulfilled by the latter. This way, on logical grounds the cofactor is given almost a residual meaning and the initial small symmetry breaking results in a great difference in the roles assigned to the price and quantity indices. Most important, in general, few axiomatic properties of the price index are inherited by its cofactor (e.g. commensurability and basis reversibility), so even if a price index satisfies many axioms, its cofactor could be very unsatisfactory from an axiomatic point of view. It is noticeable that among the axioms that a cofactor could violate, even if the price index does indeed satisfy them, there are some essential requirements, such as proportionality and homogeneity. Using a quantity index which is not proportional (or homogeneous) directly raises the question of the meaning of the quantity comparison, since the lack of proportionality or homogeneity is a serious drawback, in an axiomatic setting. For example, how can a quantity comparison be interpreted if, say, the quantity doubles over time, but the quantity index does not reveal it? Even if it is the price comparison with which one is primarily concerned, when a price index is chosen, implicitly a quantity comparison is also performed. But if the latter is not axiomatically consistent, the consistency of the former (and the consistency of the value index decomposition) could perhaps also be argued. Thus, as far as the joint consistency of price and quantity comparisons is concerned, the axiomatic properties of the cofactor of a price index should be taken into account, when the price index is being chosen. As stated above, usually this is not the case, and also from a theoretical point of view, little attention seems to have been paid to this kind of problem.

In the axiomatic framework, as it is usually presented, only the fulfillment of the factor reversibility axiom ensures that the cofactor will satisfy (with respect to quantities) all the properties fulfilled by the price index itself (with respect to prices), since in that case the cofactor and the price index share the same functional form. Nevertheless, the factor reversibility axiom is a very strong requirement and some argue that it is not an essential property (IMF, 2004); moreover, most of the indices currently used or studied do not fulfill it. In order to ensure the joint consistency of both price and quantity comparisons, a possible alternative to restricting ourselves to factor reversible price indices is to search for a class of price index formulas satisfying at least a subset of fundamental axiomatic properties *together with* their cofactors. This way, even if some priority is given to the price index, the properties of the cofactor are *ab initio* explicitly taken into account. Motivated by the

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search for such a class, the geo-logarithmic family of price indices was proposed in the early 1990s, by the Italian statistician Marco Martini (Martini, 1992a,b). Its peculiarity is that all of its members share the essential axiomatic properties of proportionality, commensurability and homogeneity together with their cofactors, so both prices and quantities can be jointly compared in a consistent way. But perhaps even more remarkably, among price indices studied in the literature and applied in practice, the only ones sharing these three axiomatic properties with their cofactors belong to the geo-logarithmic family or can be obtained from geo-logarithmic indices, by means of simple transformations, e.g. taking their basis or factor antitheses or their crossings (Martini, 2003). Thus, the geo-logarithmic family seems to have a peculiar role, when one is interested in the joint consistency of price and quantity comparisons. Notwithstanding this fact, a comprehensive study of its properties is still lacking, and the present paper aims at filling this gap; at the same time, it gives a review of already existing results and fixes some mistakes that can be found in other references. The paper is organized as follows. Section 2 gives a brief review of the axiomatic approach to price indices; Section 3 provides the formal definition of the geo-logarithmic family; Sections 4 and 5 discuss the properties of geo-logarithmic price indices; Section 6 discusses the properties of the cofactors of geo-logarithmic price indices; Section 7 studies the relationship between the geo-logarithmic family and the family of Diewert's superlative price indices; Section 8 provides some final comments and points out some open issues, needing further research; Appendices A–F collect some computational details needed in the paper.

2. Elements of axiomatic index number theory

Let $\boldsymbol{p}_a$, $\boldsymbol{p}_b$, $\boldsymbol{q}_a$ and $\boldsymbol{q}_b$ be four n-dimensional vectors ($n \geq 2$) of strictly positive components, representing the prices and the quantities of the same n goods in situation a and in situation b (b will be always considered as the *basis*, i.e. the reference situation, for the comparison). We indicate with p_{ai}, p_{bi}, q_{ai} and q_{bi} ($i = 1, \ldots, n$) the i-th component of $\boldsymbol{p}_a$, $\boldsymbol{p}_b$, $\boldsymbol{q}_a$ and $\boldsymbol{q}_b$. The ratio

$$V = \frac{\sum_{i=1}^{n} p_{ai} q_{ai}}{\sum_{i=1}^{n} p_{bi} q_{bi}} \tag{1}$$

is called the *value index* between a and b. The aim of price and quantity index theory is to decompose the value index as the product of two strictly positive functions

$$V = P(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b) \cdot Q(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b) \tag{2}$$

where P accounts for the variation of the prices and Q accounts for the variation of the quantities between situations a and b. In the axiomatic setting, the basic idea is to identify a set of axioms that a function of prices and quantities has to satisfy in order to be accepted as a price or a quantity index, and then to derive or look for formulas satisfying them.

2.1. The axioms

The literature on axiomatic index number theory is very wide (Eichhorn and Voeller, 1976a,b, 1990; Balk, 1995; Krstcha, 1988) but there is no universal agreement on the axiomatic properties needed for a formula to be considered as an index (IMF, 2004). For this reason, in this paper we focus on a set of six axioms that are generally accepted, even if some authors do in fact criticize some of them (particularly, axioms (IV) and (VI) (Diewert, 1992; Reinsdorf and Dorfman, 1999; IMF, 2004)). The list is given below.

(I) *Proportionality axiom*. Let α be a strictly positive real number; then

$$P(\alpha \boldsymbol{p}_b, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b) = \alpha.$$

(II) *Commensurability axiom*. Let U be an $n \times n$ diagonal matrix of strictly positive weights; then

$$P(U\boldsymbol{p}_a, U\boldsymbol{p}_b, U^{-1}\boldsymbol{q}_a, U^{-1}\boldsymbol{q}_b) = P(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b).$$

(III) *Homogeneity axiom*. Let α and β be two strictly positive real numbers; then

$$P(\alpha \boldsymbol{p}_a, \beta \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b) = \frac{\alpha}{\beta} P(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b).$$

(IV) *Monotonicity axiom*. Let $\hat{\boldsymbol{p}}_a > \boldsymbol{p}_a$; then

$$P(\hat{\boldsymbol{p}}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b) \geq P(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b);$$

similarly, let $\hat{\boldsymbol{p}}_b > \boldsymbol{p}_b$; then

$$P(\boldsymbol{p}_a, \hat{\boldsymbol{p}}_b, \boldsymbol{q}_a, \boldsymbol{q}_b) \leq P(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b)$$

where $\hat{\boldsymbol{p}}_s > \boldsymbol{p}_s$ if $\hat{p}_{si} \geq p_{si}$ for all $i = 1, \ldots, n$ and j exists such that $\hat{p}_{sj} > p_{sj}$, with $s = a, b$.

(V) *Basis reversibility axiom*. Let the situations a and b be exchanged and let a be taken as the basis for the comparison; then

$$P(\boldsymbol{p}_b, \boldsymbol{p}_a, \boldsymbol{q}_b, \boldsymbol{q}_a) = \frac{1}{P(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b)}.$$

(VI) *Factor reversibility axiom*. Let the vectors $\boldsymbol{p}_a$ and $\boldsymbol{p}_b$ be exchanged with the vectors $\boldsymbol{q}_a$ and $\boldsymbol{q}_b$, respectively; then

$$P(\boldsymbol{q}_a, \boldsymbol{q}_b, \boldsymbol{p}_a, \boldsymbol{p}_b) = \frac{V}{P(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b)}$$

(recall that the function $P(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b)$ is asked to be strictly positive).

An analogous axiomatic system holds for quantity indices. It is obtained simply by exchanging the role of the price and quantity vectors in the above list of axioms (I)–(VI). In general, price index formulas used for concrete price comparisons do not fulfill all the axioms listed above (Fisher, 1922; Swamy, 1965; Balk, 1995). In fact, not all the axioms are given the same relevance: axioms (I)–(III) are retained as fundamental, while axioms (IV)–(VI) are treated as less relevant.

2.2. Cofactor and correspondent of a price index

Given a price index formula $P(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b)$, two candidate quantity indices can be naturally associated to it. The first is called the *cofactor* of P, and it is defined by

$$\mathit{Cof}\ P(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b) = \frac{V}{P(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b)}; \tag{3}$$

the second is called the *correspondent* of P, and it is defined by exchanging the role of the price and quantity vectors in the formula for P:

$$\mathit{Cor}\ P(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b) = P(\boldsymbol{q}_a, \boldsymbol{q}_b, \boldsymbol{p}_a, \boldsymbol{p}_b). \tag{4}$$

In general, the cofactor and correspondent of the same price index differ; they coincide if and only if the price index satisfies the factor reversibility axiom. By virtue of (2), when a price index is chosen, its cofactor is *de facto* selected as the associated quantity index (for this reason, the cofactor is also called the *implicit* quantity index). As a consequence, a function $P(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b)$ should not be accepted as a price index, irrespective of the axiomatic properties satisfied by its cofactor. In fact, as briefly discussed in the Introduction, counterexamples can be given showing that the fundamental properties of proportionality and homogeneity of the cofactor are independent from the analogous properties of the price index (Martini, 1992a).